MQCD, ('Barely') G_2 Manifolds and (Orientifold of) a Compact Calabi-Yau [1]

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Abstract

We begin with a discussion on two apparently disconnected topics - one related to nonperturbative superpotential generated from wrapping an $M2$-brane around a supersymmetric three cycle embedded in a G_2-manifold evaluated by the path-integral inside a path-integral approach of [1], and the other centered around the compact Calabi-Yau $CY_3(3, 243)$ expressed as a blow-up of a degree-24 Fermat hypersurface in $\mathbf{WCP}^4[1, 1, 2, 8, 12]$. For the former, we compare the results with the ones of Witten on heterotic world-sheet instantons [2]. The subtopics covered in the latter include an $\mathcal{N} = 1$ triality between Heterotic, M- and F-theories, evaluation of $\mathbf{RP}^2$-instanton superpotential, Picard-Fuchs equation for the mirror Landau-Ginsburg model corresponding to $CY_3(3, 243)$, $D = 11$ supergravity corresponding to M-theory compactified on a 'barely' G_2 manifold involving $CY_3(3, 243)$ and a conjecture related to the action of antiholomorphic involution on period integrals. We then shown an indirect connection between the two topics by showing a connection between each one of the two and Witten's MQCD[3]. As an aside, we show that in the limit of vanishing "ζ", a complex constant that appears in the Riemann surfaces relevant to definining the boundary conditions for the domain wall in MQCD, the infinite series of [4] used to represent a suitable embedding of a supersymmetric 3-cycle in a G_2-mannifold, can be summed.

1 Introduction

Nonperturbative aspects of string theory have continued to be an extremely active area of work that bring about a very interesting interplay of various topics in field theory and algebraic geometry. We will be concentrating on two such topics - membrane instanton superpotentials in M theory compactified on G_2-manifolds[5], and aspects of string and M-theory compactifications on manifolds involving the compact Calabi-Yau $CY_3(3, 243)$ [6, 7, 8]. We then attempt to establish an indirect connection between these two by showing a connection between both and Witten's MQCD[3], individually.

In Section **1**, we discuss evaluation of membrane instanton superpotential and the comparison with Witten's heterotic world-sheet instanton superpotential [2]. In Section **2**, we begin with a discussion on an $\mathcal{N} = 1, D = 4$ Heterotic/M/F triality, followed by the Picard-Fuchs equation derived and solved for

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the mirror Landau-Ginsburg model for type IIA compactified on $CY_3(3, 243)$, as well as a discussion on unoriented world-sheet instanton superpotential, $D = 11$ supergravity corresponding to M theory compactified on the 'barely' G_2-manifold $\frac{CY_3(3,243)\times S^1}{\mathbf{Z}_2}$ and finally a conjecture related to a freely acting antiholomorphic involution on the period integrals for $CY_3(3, 243)$. In Section **3**, we discuss the connection between sections **2** and **3** and Witten's MQCD, individually. Section **4** has the conclusions.

2 Evaluation of the membrane instanton contribution to the superpotential

String and M theories on manifolds with G_2 and $Spin(7)$ holonomies have become an active area of research, after construction of explicit examples of such manifolds by Joyce[9]. Some explicit metrics of noncompact manifolds with the above-mentioned exceptional holonomy groups have been constructed [10].

Gopakumar and Vafa in [11], had conjectured that similar to the large N Chern-Simons/open topological string theory duality of Witten, large N Chern-Simons on S^3 is dual to closed type-A topological string theory on an S^2-resolved conifold geometry. This conjecture was verified for arbitrary genus g and arbitrary t'Hooft coupling. This duality was embedded by Vafa in type IIA, and was uplifted to M theory on a G_2 holonomy manifold by Atiyah, Maldacena and Vafa[12]. The G_2 holonomy manifold that was considered by Atiyah et al in [12] was a spin bundle over S^3 with the topology of $R^4 \times S^3$.

It will be interesting to be able to lift the above-mentioned Gopakumar-Vafa duality to M theory on a G_2-holonomy manifold, and in the process possibly get a formulation of a topological M-theory. As the type-A topological string theory's partition function receives contributions only from holomorphic maps from the world-sheet to the target space, and apart from constant maps, instantons fit the bill, as a first step we should look at obtaining the superpotential contribution of multiple wrappings of $M2$ branes on supersymmetric 3-cycles in a suitable G_2-holonomy manifold(membrane instantons). A sketch of the result anticipated for a single $M2$ brane wrapping an isolated supersymmetric 3-cycle, was given by Harvey-Moore. In this work, we have worked out the exact expression for the same, using techniques developed in [14] on evaluation of the nonperturbative contribution to the superpotential of open membrane instantons obtained by wrapping the $M2$ brane on an interval [0,1] times (thus converting the problem to that of a heterotic string wrapping) a holomorphic curve in a Calabi-Yau three-fold.

As given in [13], the Euclidean action for an $M2$ brane is given by the following Bergshoeff, Sezgin, Townsend action:

$$\mathcal{S}_\Sigma = \int d^3z \left[\frac{\sqrt{g}}{l_{11}^3} - \frac{i}{3!}\epsilon^{ijk}\partial_i \mathbf{Z}^M \partial_j \mathbf{Z}^N \partial_k \mathbf{Z}^P C_{MNP}(X(s), \Theta(s)) \right], \tag{1}$$

where $\mathbf{Z}$ is the map of the $M2$ brane world-volume to the the $D = 11$ target space M_{11}, both being regarded as supermanifolds. The g in (1), is defined as:

$$g_{ij} = \partial_i \mathbf{Z}^M \partial_j \mathbf{Z}^N \mathbf{E}_M^A \mathbf{E}_N^B \eta_{AB}, \tag{2}$$

where $\mathbf{E}_M^A$ is the supervielbein, given in [13]. $X(s)$ and $\Theta(s)$ are the bosonic and fermionic coordinates of $\mathbf{Z}$. After using the static gauge and κ-symmetry fixing, the physical degrees of freedom, are given

by $y^{m''}$, the section of the normal bundle to the $M2$-brane world volume, and $\Theta(s)$, section of the spinor bundle tensor product: $S(T\Sigma)\otimes S^-(N)$, where the $-$ is the negative $Spin(8)$ chirality, as under an orthogonal decomposition of $TM_{11}|_\Sigma$ in terms of tangent and normal bundles, the structure group $Spin(11)$ decomposes into $Spin(3)\times Spin(8)$.

The action in (1) needs to be expanded up to $O(\Theta^2)$, and the expression is (one has to be careful that in Euclidean $D=11$, one does not have a Majorana-Weyl spinor or a Majorana spinor) given as:

$$\begin{aligned}\mathcal{S}_\Sigma = \int_\Sigma\Biggl[&C+\frac{i}{l_{11}^3}vol(h)+\frac{\sqrt{h_{ij}}}{l_{11}^3}\Bigl(h^{ij}D_iy^{m''}D_jy^{n''}h_{m''n''}-y^{m''}\mathcal{U}_{m''n''}y^{n''}+O(y^3)\Bigr)\\&+\frac{i}{l_{11}^3}\sqrt{h_{ij}}\frac{1}{2}(\bar{\Psi}_MV^M-\bar{V}^M\Psi_M)+2\frac{\sqrt{h_{ij}}}{l_{11}^3}h^{ij}\bar{\Theta}\Gamma_iD_j\Theta+O(\Theta^3)\Biggr],\end{aligned}\tag{3}$$

where we follow the conventions of [13]: V_M being the gravitino vertex operator, Ψ being the gravitino field that enters via the supervielbein $\mathbf{E}_M^A$, $\mathcal{U}$ is a mass matrix defined in terms of the Riemann curvature tensor and the second fundamental form (See (24)).

After κ-symmetry fixing, like [13], we set $\Theta_2^{A\dot{a}}(s)$, i.e., the positive $Spin(8)$-chirality to zero, and following [14], will refer to $\Theta_1^{Aa}(s)$ as θ.

The Kaluza-Klein reduction of the $D=11$ gravitino is given by: $dx^M\Psi_M=dx^\mu\Psi_\mu+dx^m\Psi_\mu, \Psi_\mu(x,y)=\psi_\mu(x)\otimes\vartheta(y)$, $\Psi_m(x,y)=l_{11}^3\sum_{I=1}^{b_3}\omega^{(3)}_{I,mnp}(y)\Gamma^{pq}\chi^I(x)\otimes\tilde{\eta}(y)$, where we do not write the terms obtained by expanding in terms of $\{\omega^{(2)}_{I,mn}\}$, the harmonic 2-forms forming a basis for $H^2(X_{G_2},\mathbf{Z})$, as we will be interested in $M2$ branes wrapping supersymmetric 3-cycles in the G_2-holonomy manifold. For evaluating the nonperturbative contribution to the superpotential, following [13], we will evaluate the fermionic 2-point function: $\langle\chi^i(x_1^u)\chi^j(x_2^u)\rangle$ (where $x_{1,2}$ are the $\mathbf{R}^4$ coordinates and u [and later also v]$\equiv 7,8,9,10$ is [are] used to index these coordinates), and drop the interaction terms in the $D=4,\mathcal{N}=1$ supergravity action. The corresponding mass term in the supergravity action appears as $\partial_i\partial_jW$, where the derivatives are evaluated w.r.t. the complex scalar obtained by the Kaluza-Klein reduction of $C+\frac{i}{l_{11}^3}\Phi$ using harmonic three forms forming a basis for $H^3(X_{G_2},\mathbf{R})$. One then integrates twice to get the expression for the superpotential from the 2-point function.

The bosonic zero modes are the four bosonic coordinates that specify the position of the supersymmetric 3-cycle, and will be denoted by $x_0^{7,8,9,10}\equiv x_0^u$. The fermionic zero modes come from the fact that for every θ_0 that is the solution to the fermionic equation of motion, one can always shift θ_0 to $\theta_0+\theta'$, where $D_i\theta'=0$. This $\theta'=\vartheta\otimes\eta$ where ϑ is a $D=4$ Weyl spinor, and η is a covariantly constant spinor on the G_2-holonomy manifold.

After expanding the $M2$-brane action in fluctuations about solutions to the bosonic and fermionic equations of motion, one gets that: $\mathcal{S}|_\Sigma=\mathcal{S}_0^y+\mathcal{S}_0^\theta+\mathcal{S}_2^y+\mathcal{S}_2^\theta$, where $\mathcal{S}_0^y\equiv\mathcal{S}_\Sigma|_{y_0,\theta_0}$ $\mathcal{S}_0^\theta\equiv\mathcal{S}_\Sigma^\theta+\mathcal{S}_\Sigma^{\theta^2}|_{y_0,\theta_0}$; $\mathcal{S}_2^y\equiv\frac{\delta^2\mathcal{S}_\Sigma}{\delta y^2}|_{y_0,\theta_0=0}(\delta y)^2$; $\mathcal{S}_2^\theta\equiv\frac{\delta^2\mathcal{S}}{\delta\theta^2}|_{y_0,\theta_0=0}(\delta\theta)^2$. Following [14], we consider classical values of coefficients of $(\delta y)^2,(\delta\theta)^2$ terms, as fluctuations are considered to be of $\mathcal{O}(\sqrt{\alpha'})$.

Now,

$$\begin{aligned}&\langle\chi^i(x_1^u)\chi^j(x_2^u)\rangle=\\&\int\mathcal{D}\chi e^{K.E\ of\ \chi}\chi^i(x)\chi^j(x)\int d^4x_0e^{-\mathcal{S}_0^y}\end{aligned}$$

$$\times \int d\vartheta^1 d\vartheta^2 e^{-\mathcal{S}_0^\theta} \int \mathcal{D}\delta y^{m''} e^{-\mathcal{S}_2^y} \int \mathcal{D}\delta\bar{\theta}\mathcal{D}\delta\theta e^{-\mathcal{S}_2^\theta}. \tag{4}$$

We now evaluate the various integrals that appear in (4) above starting with $\int d^4 x e^{-\mathcal{S}_0^y}$:

$$\int d^4 x_0 e^{-\mathcal{S}_0^y} = \int d^4 x_0 e^{[iC - \frac{1}{l_{11}^3} vol(h)]}. \tag{5}$$

Using the 11-dimensional Euclidean representation of the gamma matrices as given in [13], $\mathcal{S}_0^\theta + \mathcal{S}_0^{\theta^2}|_\Sigma = \frac{i}{2l_{11}^3}\int_\Sigma \sqrt{h_{ij}}\bar{\Psi}_M V_M d^3 s$, where using $\partial_i x_0^u = 0$, and using U to denote coordinates on the G_2-holonomy manifold, $V^U = h_{ij}\partial_i y_0^U \partial_j y^V \gamma_V \theta_0 + \frac{i}{2}\epsilon^{ijk}\partial_i y_0^U \partial_j y_0^V \partial_k y_0^W \Gamma_{VW}\theta_0$,

$$\int d\vartheta_1 d\vartheta_2 e^{\frac{i}{2l_{11}^3}\sum_{I=1}^{b_3}\sum_{\alpha=1}^{2}\sum_{i=1}^{8}(\bar{\chi}(x)\sigma^{(i)})_\alpha \vartheta_\alpha \omega_I^{(i)}} = -\frac{1}{4l_{11}^3}\sum_{I=1}^{b_3}\sum_{i<j=1}^{8}\omega_I^{(i)}\omega_I^{(j)}(\bar{\chi}\sigma^{(i)})_1(\bar{\chi}\sigma^{(j)})_2, \tag{6}$$

where one uses that for G_2-spinors, the only non-zero bilinears are: $\eta^\dagger\Gamma_{i_1\ldots i_p}\eta$ for $p = 0(equiv$ constant), $p = 3(\equiv$ calibration 3-form), $p = 4(\equiv)$ Hodge dual of the calibration 3-form and $p = 7(\equiv$ volume form). We follow the following notations for coordinates: u, v are $\mathbf{R}^4$ coordinates, $\hat{U}, \hat{V}$ are G_2-holonomy manifold coordinates that are orthogonal to the $M2$ world volume (that wraps a supersymmetric 3-cycle embedded in the G_2-holonomy manifold), and U, V are G_2-holonomy manifold coordinates. The tangent/curved space coordinates for Σ are represented by a'/m' and those for $X_{G_2}\times\mathbf{R}^4$ are represented by a''/m''.

We now come to the evaluation of $\mathcal{S}_2^\theta|_{y_0,\theta_0=0}$. Using the equality of the two $O((\delta)\Theta^2)$ terms in the action of Harvey and Moore, and arguments similar to the ones in [14], one can show that one needs to evaluate the following bilinears: $\delta\bar{\Theta}\Gamma_{a'}\partial_i\delta\Theta$, $\delta\bar{\Theta}\Gamma_{a''}\partial_i\delta\Theta$, $\delta\bar{\Theta}\Gamma_{a'}\Gamma_{AB}\delta\Theta$, and $\delta\bar{\Theta}\Gamma_{m''}\Gamma_{AB}\delta\Theta$. Evaluating them, one gets:

$$\begin{aligned}
&\mathcal{S}_2^\theta|_{y_0,\theta_0=0} \equiv \int_\Sigma d^3 s\,\delta\theta^\dagger\mathcal{O}_3\delta\theta,\\
&\text{where } \mathcal{O}_3 \equiv\\
&\frac{2i}{l_{11}^3}\sqrt{h_{ij}}\Biggl[h^{ij}\delta_j^{m'}\Biggl(-e_{m'}^3\sigma^3\otimes\mathbf{1}_8\partial_i - 2i[e_{m'}^1\omega_i^{23} + e_{m'}^2\omega_i^{31} + e_i^3\omega_i^{12}]\sigma^3\otimes\mathbf{1}_8\\
&-2[e_{m'}^1\omega_i^{13} + e_{m'}^2\omega_i^{23}]\sigma^3\otimes\mathbf{1}_8 + e_{m'}^3\frac{\omega^{b''c''}}{2}\sigma^3\otimes\gamma_{b'c'}\Biggr)\\
&+ih^{ij}e_{m'}^{a'}\omega_i^{b'c''}\sigma^2\otimes\gamma_{c''}[\delta^{a'b'} + i\epsilon^{a'b'c'}\delta_3^{c'}]\\
&+ih^{ij}\partial_j y^{m''}e_{m''}^{a''}[\sigma^2\otimes\gamma_{a''}\partial_i - (\omega_i^{b''c''}\sigma^2\frac{1}{6}\otimes\gamma_{a''b''c''} - \frac{1}{2}\omega^{b'c''}\delta_3^{b'}\sigma^3\otimes\gamma_{a''c''})]\Biggr],
\end{aligned} \tag{7}$$

Hence, the integral over the fluctuations in θ will give a factor of $\sqrt{det\mathcal{O}_3}$ in Euclidean space.

The expression for $\mathcal{S}_2^y|_{\Sigma_{y_0,\theta_0=0}}$ is identical to the one given in [14], and will contribute $\frac{1}{\sqrt{det\mathcal{O}_1 det\mathcal{O}_2}}$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are as given in the same paper:

$$\begin{aligned}
&\mathcal{O}_1 \equiv \eta_{uv}\sqrt{g}g^{ij}\mathcal{D}_i\partial_j\\
&\mathcal{O}_2 \equiv \sqrt{g}(g^{ij}\mathcal{D}_i h_{\hat{U}\hat{V}} D_j + \mathcal{U}_{\hat{U}\hat{V}}).
\end{aligned} \tag{8}$$

The mass matrix $\mathcal{U}$ is expressed in terms of the curvature tensor and product of two second fundamental forms. $\mathcal{D}_i$ is a covariant derivative with indices in the corresponding spin-connection of the type $(\omega_i)^{m''}_{n''}$ and $(\omega_i)^{m'}_{n'}$, and D_i is a covariant derivative with corresponding spin connection indices of the former type.

Hence, modulo supergravity determinants, and the contribution from the fermionic zero modes, the exact form of the superpotential contribution coming from a single $M2$ brane wrapping an isolated supersymmetric cycle of G_2-holonomy manifold, is given by:

$$\Delta W = e^{iC - \frac{1}{l^3_{11}} vol(h)} \sqrt{\frac{det O_3}{det\mathcal{O}_1 \ det\mathcal{O}_2}}. \tag{9}$$

We do not bother about 5-brane instantons, as we assume that there are no supersymmetric 6-cycles in the G_2-holonomy manifold that we consider.

In [13], it is argued that for an "associative" 3-fold Σ in the G_2-holonomy manifold, the structure group $Spin(8)$ decomposes into $Spin(4)_{\mathbf{R^4}} \times Spin(4)_{X_{G_2} \backslash \Sigma}$. After gauge-fixing under κ-symmetry,

$$\Theta = \Big((\Theta_{--})^{AY}_{\alpha}, (\Theta_{++})^{\dot{Y}}_{\dot{\alpha} A}; 0, 0 \Big), \tag{10}$$

where $A, \overset{(\cdot)}{\alpha}, \overset{(\cdot)}{Y}$ are the $Spin(3), Spin(4)_{\mathbf{R^4}}, Spin(4)_{X_{G_2} \backslash \Sigma}$ indices respectively. The G_2 structure allows one to trade off $(\Theta_{--})^{AY}_{\alpha}$ for fermionic 0- and 1-forms: η, χ_i, which together with $y^u \equiv y^{\alpha\dot{\alpha}}$, form the Rozansky-Witten(RW) multiplet. Similarly, $(\Theta_{++})^{\dot{Y}}_{\dot{\alpha} A}$ gives the Mclean multiplet: $(y^{A\dot{y}}, \nu^{\dot{Y}}_{\dot{\alpha} A})$. The RW model is a $D = 3$ topological sigma model on a manifold embedded in a hyper-Kähler manifold X_{4n} [15]. If $\phi^{M(=1,...,4n)}(x^i)$ are functions from mapping M to X, then the RW action is given by:

$$\int_\Sigma \sqrt{h_{ij}} \left[\frac{1}{2} h_{MN} \partial_i \phi^M \partial_j \phi^N h^{ij} + \epsilon_{IJ} h^{ij} \chi^I_i D_j \eta^J + \frac{1}{2\sqrt{h_{ij}}} \epsilon^{ijk} \left(\epsilon_{IJ} \chi^I_i D_j \chi^J_k + \frac{1}{3} \Omega_{IJKL} \chi^I_i \chi^J_j \chi^L_k \eta^L \right) \right], \tag{11}$$

where $\Omega_{IJKL} = \Omega_{JIKL} = \Omega_{IJLK}$. Then, dropping the term proportional to Ω_{IJKL}, one sees that the terms in (3), are very likely to give the RW action in (11). In [13], $n = 1$.

As the RW and Mclean's multiplets are both contained in $\delta\theta$, hence (using the notations of [13]) $det'(L_-)det'(\not{D}_E)$ will be given by $det\mathcal{O}_3$ - it is however difficult to disentangle the two contributions. The relationship involving the spin connections on the tangent bundle and normal bundle (the anti self-dual part of the latter) as given in [13], can be used to reduce the number of independent components of the spin connection and thus simplify (7). Further, $(det'\Delta_0)^2 |det'(\not{D}_E)|$ should be related to $\sqrt{det\mathcal{O}_1 det\mathcal{O}_2}$. Hence, the order of $H_1(\Sigma, \mathbf{Z})$ must be expressible in terms of $\sqrt{det\mathcal{O}_{1,2,3}}$ for $M2-$brane wrapping a rigid supersymmetric 3-cycle. However, we wish to emphasize that (7), unlike the corresponding result of [13], is equally valid for $M2-$brane wrapping a non-rigid supersymmetric 3-cycle, as considered in Section 4.

We now explore the possibility of cancellations between the bosonic and fermionic determinants For bosonic determinants $detA_b$, the function that is relevant is $\zeta(s|A_b)$, and that for fermionic determinants $detA_f$, the function that is additionally relevant is $\eta(s|A_f)$. The integral representation of the former involves $Tr(e^{-tA_b})$, while that for the latter involves $Tr(Ae^{-tA^2})$ (See [16]):

$$\zeta(s|A_b) = \frac{1}{\Gamma(2s)} \int_0^\infty dt t^{s-1} Tr(e^{-tA_b}); \eta(s|A_f) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty dt t^{\frac{s+1}{2}} Tr(A_f e^{-tA^2_F}), \tag{12}$$

where to get the UV-divergent contributions, one looks at the $t \to 0$ limit of the two terms. To be more precise (See [17])

$$\begin{aligned}
& lndetA_b = -\frac{d}{ds}\zeta(s|A_b)|_{s=0} \\
& = -\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty dtt^{s-1}Tr(e^{-tA_b})\right)|_{s=0}; \\
& lndetA_f = -\frac{1}{2}\frac{d}{ds}\zeta(s|A_f^2)|_{s=0} \mp \frac{i\pi}{2}\eta(s|A_f)|_{s=0} \pm \frac{i\pi}{2}\zeta(s|A_f^2)|_{s=0} \\
& = \left[-\frac{1}{2}\frac{d}{ds} \pm \frac{i\pi}{2}\right]\left(\frac{1}{\Gamma(s)}\int_0^\infty dtt^{s-1}Tr(e^{-tA_f^2})\right)|_{s=0} \mp \frac{i\pi}{2}\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty dtt^{\frac{s+1}{2}-1}Tr(A_f e^{-tA_f^2})|_{s=0},
\end{aligned} \tag{13}$$

where the $\mp$ sign in front of $\eta(0)$, a non-local object, represents an ambiguity in the definition of the determinant. The $\zeta(0|A_f^2)$ term can be reabsorbed into the contribution of $\zeta'(0|A_f^2)$, and hence will be dropped below. Here $Tr \equiv \int dx\langle x|...|x\rangle \equiv \int dxtr(...)$. The idea is that if one gets a match in the Seeley - de Witt coefficients for the bosonic and fermionic determinants, implying equality of UV-divergence, this is indicative of a possible complete cancellation.

The heat kernel expansions for the bosonic and fermionic determinants[18] are given by:

$$tr(e^{-tA_b}) = \sum_{n=0}^\infty e_n(x, A_b)t^{\frac{(n-m)}{2}}, tr(A_f e^{-tA_f^2}) = \sum_{n=0}^\infty a_n(x, A_f)t^{\frac{(n-m-1)}{2}}, \tag{14}$$

where for m is the dimensionality of the space-time. For our case, we have a compact 3-manifold, for which $e_{2p+1} = 0$ and $a_{2p} = 0$. For Laplace-type operators A_b, and Dirac-type operators A_f, the non-zero coefficients are determined to be the following:

$$e_0(x, A_b) = (4\pi)^{-\frac{3}{2}}Id,\ \ e_2(x, A_b) = (4\pi)^{-\frac{3}{2}}\left[\alpha_1 E + \alpha_2\tau Id\right], \tag{15}$$

where α_i's are constants, $\tau \equiv R_{ijji}$, and Id is the identity that figures with the scalar leading symbol in the Laplace-type operator A_b (See [18]), and

$$\begin{aligned}
& E \equiv B - G^{ij}(\partial_i\omega_j + \omega_i\omega_j - \omega_k\Gamma^k_{ij}), \\
& A_b \equiv -(G^{ij}Id\partial_i\partial_j + A^i\partial_i + B), \\
& \omega_i = \frac{G_{ij}(a^j + G^{kl}\Gamma^j_{kl}Id)}{2}.
\end{aligned} \tag{16}$$

To actually evaluate e_0 and e_2, we need to find an example of a regular G_2-holonomy manifold that is metrically $\Sigma \times M_4$, where Σ is a supersymmetric 3-cycle on which we wrap an $M2$ brane once, and M_4 is a four manifold. One such example was obtained in [19], that be regarded as a cone over a base $S^3 \times R^3$, that I was referring to is actually:

$$ds^2 = dr^2 + \frac{1}{y}\sum_{j=1}^3(x^2\sigma_j^2 + mn\sigma_j d\theta_j - mxd\theta_j^2), \tag{17}$$

where $\alpha_{1,2,3}$ are the left-invariant $SU(2)$ 1-forms, m, n are two parameters that characterize $H^3(S^3 \times R^3, \mathbf{R}) = \mathbf{R} \oplus \mathbf{R}$, $x = -\frac{m^{\frac{1}{3}}}{4}(r-r_0)^2$, $y = \frac{m^{\frac{2}{3}}}{4}(r-r_0)^2$, r_0 being an integration constant that for convenience can be set to zero.

Now take the simplifying limit $n = 0$. This for $m = 1$ gives:

$$ds_7^2 = dr^2 + \frac{r^2}{4}\sum_{i=1}^{3}\sigma_i^2 + \sum_{i=1}^{3} d\theta_i^2, \tag{18}$$

where σ_i's are left-invariant one-forms obeying the SU(2) algebra: $d\sigma_i = -\frac{1}{2}\epsilon^{ijk} d\sigma^j \wedge d\sigma^k$, given by:

$$\sigma_1 \equiv cos\psi d\theta + sin\psi sin\theta d\phi, \sigma_2 \equiv -sin\psi d\theta + cos\psi sin\theta d\phi, \sigma_3 \equiv d\psi + cos\theta d\phi. \tag{19}$$

The metric in (refeq:sigmas) does not have a G_2 holonomy. It is argued that (18) is what (17) asymptotes to, for $n \neq 0$. The heat-kernel asymptotics analysis below, can either be treated as one for membrane instanton superpotentials for non-compact G_2 manifolds IN THE LARGE DISTANCE-LIMIT($\leftrightarrow r \rightarrow \infty$), or equivalently for a non-compact M_7 with a supersymmetric 3-cycle(a T^3) embedded in it that nevertheless gives $\mathcal{N} = 1, D = 4$ supersymmetry.

To see that the T^3 corresponds to a supersymmetric 3-cycle, we need to show that the pull-back of the calibration Φ_3 restricted to Σ, is the volume form on Σ (See [20]). Φ_3 using the notations of [19] is given by:

$$\Phi_3 = e^{125} + e^{147} + e^{156} - e^{246} + e^{237} + e^{345} + e^{567}, \tag{20}$$

where $e^{ijk} \equiv e^i \wedge e^j \wedge e^k$. Let $1, ..., 7$ denote $r, \psi, \theta, \phi, \theta_1, \theta_2, \theta_3$. Hence, when restricted to $\Sigma(\theta_1, \theta_2, \theta_3)$ using the static gauge, one gets: $\Phi_3|_\Sigma = e^{567} = d\theta_1 \wedge d\theta_2 \wedge d\theta_3$, which is the volume form on T^3. Thus, the T^3 of (18) is a supersymmetric 3-cycle.

For (18), one sees that $g_{ij} = \delta_{ij} + \partial_i y_0^{\hat{U}} \partial_j y_0^{\hat{V}} g_{\hat{U}\hat{V}}$, having used the definition of g_{ij} as a pull-back of the space-time metric g_{MN}, static gauge and that $\partial_i y_0^u = 0$. If one assumes that the coordinates r, ψ, θ, ϕ are very slowly varying functions of $\theta_1, \theta_2, \theta_3$, one sees that $g_{ij} \sim \delta_{ij}$. This simplifies the algebra, though one can work to any desired order in $(\partial_i y_0^{\hat{U}})^{p(>0)}$, and get conclusions similar to the ones obtained below.

Lets first consider the Seeley de-Wit coefficients for $\mathcal{O}_1$. Now, in the above adiabatic approximation, the world volume metric of the $M2$-brane is flat. Hence, the Christoffel connection Γ^i_{jk} for $\mathcal{O}_1$, vanishes. Now, $\omega_i^{a'b'} \sim \delta_i^{m'} \omega_{m'}^{a'b'}$, where

$$\omega^{ab}_{m'} = e^{[a}{}_{n'} g^{n'l'} (\partial_{m'} e_{l'}{}^{b']} - \Gamma^{p'}_{l'm'} e_{p'}{}^{b']}), \tag{21}$$

the antisymmetry indicated on the right hand side of (21) being applicable only to the tangent-space indices a', b', and where for (18), the following are the non-zero vielbeins:

$$\begin{aligned}
& e^1{}_r = 1; e^2{}_\theta = \frac{r}{2} cos\psi,\ e^2{}_\phi = \frac{r}{2} sin\psi sin\theta; \\
& e^3{}_\theta = -\frac{r}{2} sin\psi,\ e^3{}_\phi = cos\psi sin\theta; e^4{}_\psi = \frac{r}{2},\ e^4{}_\phi = \frac{r}{2} cos\theta; \\
& e^5{}_{\theta_1} = e^6{}_{\theta_2} = e^7{}_{\theta_3} = 1.
\end{aligned} \tag{22}$$

Hence, for the G_2 metric of (18), $\partial_{m'} e_{l'}{}^{b'} = 0$. Also, $\Gamma^{p'}_{l'm'} = 0$. Thus, $\omega_i^{\mathcal{D}} \sim 0$.

In the adiabatic approximation, $\tau \sim 0$.

Hence,

$$e_0(x, \mathcal{O}_1) = (4\pi)^{-\frac{3}{2}}; e_2(x, \mathcal{O}_1) = 0. \tag{23}$$

We next consider evaluation of $e_{0,2}(x, \mathcal{O}_2)$. Once again, the Christoffel connection Γ^i_{jk} vanishes. Again, $\omega^{D,\mathcal{D}} \sim 0$. Also, $\partial_i h_{\hat{U}\hat{V}} \sim 0$. Hence, $A^i \sim 0$.

Now,

$$\mathcal{U}_{\hat{U}\hat{V}} \equiv \frac{1}{2} R^{m'}_{\hat{U}m'\hat{V}} + \frac{1}{8} Q^{m'n'}_{\hat{U}} Q_{m'n'\hat{V}}, \tag{24}$$

where the second fundamental form is defined via:

$$\Gamma^{m''}_{k'l'} \equiv -\frac{1}{2} g^{m''n''} Q_{k'l'n''}. \tag{25}$$

Using:

$$R^{m'}_{\hat{U}m'\hat{V}} = \partial_{\hat{V}} \Gamma^{m'}_{\hat{U}m'} - \partial_{m'} \Gamma^{m'}_{\hat{U}\hat{V}} + \Gamma^V_{\hat{U}m'} \Gamma^{m'}_{V\hat{V}} - \Gamma^V_{\hat{U}\hat{V}} \Gamma^{m'}_{Vm'}, \tag{26}$$

and the fact that the non-zero Christoffel symbols do not involve m' as one of the (three) indices and that their values are m'-independent, one sees that

$$\mathcal{U}_{\hat{U}\hat{V}} = 0. \tag{27}$$

Hence, $B \sim 0$.

For evaluating $\tau \equiv g^{i_1 i_2} g^{j_1 j_2} R_{i_1 j_1 j_2 i_2} = g^{i_1 i_2} g^{j_1 j_2} g_{i_1 l_1} R^{l_1}_{j_1 j_2 i_2}$, one needs to evaluate $R^{l_1}_{j_1 j_2 i_2} = \partial_{i_2} \Gamma^{l_1}_{j_1 j_2}$ -$\partial_{j_2} \Gamma^{l_1}_{j_1 i_2}$ +$\Gamma^p_{j_1 j_2} \Gamma^l_{p i_2}$ -$\Gamma^p_{j_1 i_2} \Gamma^{l_1}_{p j_2}$. This will be evaluated using the metric given by $G^{ij} = g^{ij} \sqrt{g} h_{\hat{U}\hat{V}}$, where we will use the adiabatic approximation: $g_{ij} \sim \delta^{m'}_i \delta^{n'}_j g_{m'n'}$. Due to the $\hat{U}\hat{V}$ indices, the Ricci scalar τ is actually a matrix in the $X_{G_2} \setminus \Sigma$ space. In the adiabatic approximation, only the product of the two Christoffel symbols in $R^{l_1}_{j_1 j_2 i_2}$ is non-zero, and is given by the following expression:

$$\begin{aligned}
&\Gamma^{p''}_{j_1 j_2} \Gamma^{l_1}_{p'' i_2} - \Gamma^{p''}_{j_2 i_1} \Gamma^{l_1}_{p'' j_2} \\
&= -\frac{h_{\hat{U}\hat{V}}}{4} \delta_{j_1 j_2} \delta^{l_1}_{i_2} \left[\left(\partial_r \left[\frac{1}{h_{\hat{U}\hat{V}}} \right] \right)^2 + \frac{4}{r^2} \left(\partial_\theta \left[\frac{1}{h_{\hat{U}\hat{V}}} \right] \right)^2 \right] \\
&= -\frac{\delta_{j_1 j_2} \delta^{l_1}_{i_2}}{4} \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \frac{16}{r^4} & 0 & \frac{16}{r^4 cos\theta} + \frac{4 sin^2\theta}{r^2 cos^3\theta} \\ 0 & 0 & \frac{16}{r^4} & 0 \\ 0 & \frac{16}{r^4 cos\theta} + \frac{4 sin^2\theta}{r^2 cos^3\theta} & 0 & \frac{16}{r^4} \end{array} \right).
\end{aligned} \tag{28}$$

Hence, on taking $tr_{X_{G_2} \setminus \Sigma}$, one gets:

$$\tau = \frac{72}{r^4}. \tag{29}$$

Hence,

$$e_0(x, \mathcal{O}_2) = (4\pi)^{-\frac{3}{2}};\ e_2(x, \mathcal{O}_2) = (4\pi)^{-\frac{3}{2}} \frac{72\alpha_2}{r^4}. \tag{30}$$

We now do a heat-kernel asymptotics analysis of the fermionic determinant $det\mathcal{O}_3$. The fermionic operator $\mathcal{O}_3$ can be expressed as:

$$\mathcal{O}_3 \equiv \sqrt{h}h^{ij}\Gamma_j D_i = \sqrt{h}h^{ij}\Gamma_j\left(\partial_i + \frac{1}{4}\omega_i^{a'b'}\Gamma_{a'b'} + \frac{1}{4}\omega_i^{a'b''}\Gamma_{a'b''}\right) \equiv G^{ij}\Gamma_j\partial_i - r, \tag{31}$$

where

$$G^{ij} \equiv \sqrt{h}h^{ij}; r \equiv \frac{-1}{4}\sqrt{h}h^{ij}\Gamma_j\left(\omega_i^{a'b'}\Gamma_{a'b'} + \omega_i^{a'b''}\Gamma_{a'b''}\right). \tag{32}$$

$\mathcal{O}_3$ is of the Dirac-type as $\mathcal{O}_3^2$ is of the Laplace-type, as can be seen from the following:

$$\begin{aligned}
&\mathcal{O}_3^2 \equiv G^{ij}\partial_i\partial_j + A^i\partial_i + B, \text{ where :}\\
&G^{ij} \equiv hh^{ij};\\
&A^i \equiv \sqrt{h}\Gamma^j\Gamma_k\partial_j(\sqrt{h}h^{kl}) + 2h\Gamma^i\Gamma^l\omega_l^{CD}\Gamma_{CD};\\
&B \equiv \sqrt{h}\Gamma^j\Gamma_k\Gamma_{CD}\partial_j(\sqrt{h}h^{kl}\omega_l^{CD}) + h\Gamma^j\Gamma^l\omega_j^{AB}\Gamma_{AB}\omega_l^{CD}\Gamma_{CD}.
\end{aligned} \tag{33}$$

Now,

$$\mathcal{O}_3 \equiv G^{ij}\Gamma_j \bigtriangledown_i - \phi, \tag{34}$$

where $\phi \equiv r + \Gamma^i\omega_i$, and

$$\begin{aligned}
\omega_l &\equiv \frac{G_{il}}{2}(-\Gamma^j\partial_j\Gamma^i + \{r,\Gamma^i\} + G^{jk}\Gamma^i_{jk})\\
&= \frac{h_{il}}{2\sqrt{h}}\Bigg(\frac{1}{4}\sqrt{h}h^{i'j'}\{\Gamma_{j'}(\omega_{i'}^{a'b'}\Gamma_{a'b'} + \omega_{i'}^{a'b''}\Gamma_{a'b''}),\Gamma^i\}\\
&+\frac{hh^{jk}h^{ii'}}{2}\left(\partial_j\left[\frac{h_{ki'}}{\sqrt{h}}\right] + \partial_k\left[\frac{h_{ji'}}{\sqrt{h}}\right] - \partial_{i'}\left[\frac{h_{jk}}{\sqrt{h}}\right]\right)\Bigg).
\end{aligned} \tag{35}$$

The Seeley-de Witt coefficients a_i are given by (See [18]):

$$a_1(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) = -(4\pi)^{-\frac{3}{2}}tr(\phi); a_3(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) = -\frac{1}{6}(4\pi)^{-\frac{3}{2}}tr(\phi\tau + 6\phi\mathcal{E} - \Omega_{a'b';a'}\Gamma_{b'}), \tag{36}$$

where

$$\mathcal{E} \equiv -\frac{1}{2}\Gamma^i\Gamma^j\Omega_{ij} + \Gamma^i\phi_{;i} - \phi^2, \Omega_{ij} \equiv \partial_i\omega_j - \partial_j\omega_i + [\omega_i,\omega_j]. \tag{37}$$

Now, e.g., $\Gamma^i = \partial^i y^M\Gamma_M$, where $y^M \equiv y^{m',\hat{U},u}$ and given that $\partial_i y^u = 0$, then in the static gauge, $\Gamma^i = \delta^i_{m'}\Gamma_{m'} + \partial^i y^{\hat{U}}\Gamma_{\hat{U}} = \delta^i_{m'}e^{a'}_{m'}\Gamma_{a'} + \partial^i y^{\hat{U}}e^{\hat{A}}_{\hat{U}}\Gamma_{\hat{A}}$. Now, lets make the simplifying assumption as done for the bosonic operators, we assume that $y^{\hat{U}}$ varies very slowly w.r.t. the $M2$-brane world-volume coordinates. Hence, we drop all terms of the type $(\partial_i y^{\hat{U}})^{p(>0)}$. The conclusion below regarding the vanishing of the Seeley-de Witt coefficients a_1 and a_3, will still be valid. The dropping of $(\partial_i y^U)^{p(>0)}$-type terms will be indicated by $\sim$ as opposed to $=$ in the equations below. One finally gets:

$$a_1(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) = a_3(x, G^{ij}\Gamma_j \bigtriangledown_i - \phi) \sim 0. \tag{38}$$

We conjecture that in fact, $a_{2n+1}(x, G^{ij}\Gamma_j \bigtriangledown_i -\phi) \sim 0$ for $n = 0, 1, 2, 3, ...$

By using reasoning similar to the one used in Appendix A, one can show that:

$$e_0(x, \mathcal{O}_3^2) = (4\pi)^{-\frac{3}{2}}; e_2(x, \mathcal{O}_3^2) \sim 0. \tag{39}$$

From the extra factor of $\frac{1}{2}$ multiplying the $\zeta'(0|\mathcal{O}_3^2)$ relative to $\zeta'(0|\mathcal{O}_1)$ in (13), and (38) and (39), one sees the possibility that:

$$\frac{ln[det\mathcal{O}_3]}{ln[det\mathcal{O}_1]} \sim \frac{1}{2}. \tag{40}$$

In conclusion, one sees that Seeley-de Witt coefficients of the fermionic operator $\mathcal{O}_3$ are proportional to those of the bosonic operator $\mathcal{O}_1$ in the adiabatic approximation, to the order calculated, for the G_2-metric (18). This is indicative of possible cancellation between them. This is expected, as the $M2$-brane action has some supersymmetry. As $b_1(T^3) = 3 > 0$, thus the supersymmetric 3-cycle of (18) is an example of a non-rigid supersymmetric 3-cycle. The result of [13] is not applicable for this case. On the other hand, the superpotential written out as determinants, as in this work, is still valid. The corresponding modified formula in [13] might consist, as prefactors, in addition to the phase, the torsion elements of $H_1(\Sigma, \mathbf{Z})$, represented by $|H_1(\Sigma, \mathbf{Z})|'$ in [15], and perhaps a geometrical quantity that would encode the G_2-analog of the arithmetic genus condition in the context of 5-brane instantons obtained by wrapping $M5$-brane on supersymmetric 6-cycles in CY_4 in [21]. The validity of the arithmetic genus argument, even for CY_4, needs to be independently verified though.

For heterotic world sheet instantons, as studied in [2], the expression for the nonperturbative superpotential is given by:

$$\Delta W = exp[-\frac{A(C)}{2\pi\alpha'} + \int_C B]\frac{\text{Pfaff}'\partial_{S_+\otimes S_+(N)}\text{Pfaff}(\bar{\partial}_{O(-1)\otimes V}}{(det\partial_{O(-1)})^2)(\text{det}\partial_{O(0)})^2(\text{det}\bar{\partial}_{\mathcal{O}(-1)})^2(\text{det}'\bar{\partial}_{\mathcal{O}(0)})^2} \tag{41}$$

In this expression, the convention followed by Witten is that the left fermionic movers come with the kinetic operator $\bar{\partial}$ and the right movers come with the kinetic operator ∂. Further, the bosonic zero modes are associated with the left-movers corresponding to translation in $\mathbf{R}^4$, and the fermionic zero modes are associated with the right-movers. The left-moving fermions are sections: $\Gamma[S_-(C \hookrightarrow CY_3) \otimes V]$ and the right-moving fermions are sections: $\Gamma[S_+(C \hookrightarrow CY_3) \otimes S_+(N)]$, where C is the genus-zero curve around which the string world-sheet wraps around to give world-sheet instanton, V is the $SO(32)$ gauge bundle, and N is the normal bundle to C in $\mathbf{R}^4 \times CY_3$. As fiber bundles, $S_-(C) \cong \mathcal{O}(-1)$, hence the left movers are sections of $\mathcal{O}(-1) \otimes V \equiv V(-1)$. Now, $N|_{CY_3} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and $N|_{\mathbf{R}^4} \cong \mathcal{O}(0) \oplus \mathcal{O}(0)$. Also, the eight real bosons transverse to C can be combined into four complex bosons.

After the bosonic-fermionc determinant cancelation:

$$\text{Pfaff}'\partial_{S_+\otimes S_+(N)} = det\partial_{\mathcal{O}(-1)})^2(det\partial_{\mathcal{O}(0)})^2, \tag{42}$$

what survives in the heterotic world-sheet instanton superpotential is

$$\Delta W = Exp[-\frac{A(C)}{2\pi\alpha'} + i\int_C B]\frac{\text{Pfaff}(\bar{\partial}_{V(-1)})}{(\text{det}\bar{\partial}_{\mathcal{O}(-1)})^2(\text{det}'\bar{\partial}_{\mathcal{O}(0)})^2}. \tag{43}$$

Its here that the difference between the present result and the form of the result in [2] becomes manifest. In the latter, after equality of bosonic and fermionic determinants one gets Pfaff$(\bar{\partial}_{O(-1)\otimes V}$ in the numerator. As argued in [2], this expression can vanish if the gauge bundle restricted to the genus-0 curve C, is trivial. To see this, Witten argued that any $SO(32)$ gauge bundle V over a genus-zero curve C, can be written as:$V = \oplus_{i=1}^{16}\mathcal{O}(m_i)\oplus\mathcal{O}(-m_i)$, implying $V(-1) = \oplus_{i=1}^{16}\mathcal{O}(m_i-1)\oplus\mathcal{O}(-m_i-1)$. Now, dim[ker$(\bar{\partial}_{\mathcal{O}(s)}$]]$=s+1$ if $s > 0$ and zero if $s < 0$. Thus, dim[ker$(\bar{\partial}_{V(-1)})$] $= \sum_{i=1}^{16} m_i \equiv 0$ iff $m_i = 0$ for all $i = 1, ..., 16$. In our result for the membrane instanton for non-rigid supersymmetric 3-cycle, the superpotential can never vanish because of unity in the numerator.

3 String and M-Theory Compactifications involving $CY_3(3, 243)$

In this section, we will discuss two topics. One will be related to an $\mathcal{N} = 1, D = 4$ triality between Heterotic, M and F theories. The other is related to studying some algebraic geometric aspects of the compact Calabi-Yau $CY_3(3, 243)$ such as period integrals.

3.1 An $\mathcal{N} = 1$ Triality by Spectrum Matching

As $\mathcal{N} = 1$ supersymmetry in four dimensions is of phenomenological interest, it is important to understand possible dualities between different ways of arriving at the same amount of supersymmetry via suitable compactifications. In this regard, the results of [23, 22] are of particular interest. While [23] construct such string dual pairs, [22] also give $\mathcal{N} = 1$ Heterotic/M-theory dual pairs. As M-theory on G_2-holonomy manifolds gives $\mathcal{N} = 1$ supersymmetry, especially after explicit examples of the same (and Spin(7)) in [9, 10], exceptional holonomy compactifications of M-theory becomes quite relevant for the above purpose. In the literature, so far, the $\mathcal{N} = 1, D = 4$ Heterotic/M-theory dual pair constructions, stem one way or the other from the Heterotic on T^3 and M-theory on $K3$ $D = 7$ duality [24, 25]. The question is what the $\mathcal{N} = 1$ Heterotic/M-theory analog of the Heteoric/type IIA $\mathcal{N} = 1$ dual pair of [23] is. As the $D = 7$ Heterotic/M-Theory duality is related to the $D = 6$ Heterotic/String duality (as the decompactification limit - see [25]), it is reasonable to think that there has to be such an $\mathcal{N} = 1$ Heterotic/M-theory dual pair. Additionally, it will be interesting to work out an example that is able to explicitly relate an $\mathcal{N} = 1$ Heterotic theory to M and F theories, as opposed to examples in the literature on only $\mathcal{N} = 1$ Heteorotic/type IIA or Heterotic/M-theory or Heterotic/F-theory dual pairs. We propose that the M-theory side is given by a 7-manifold of $SU(3) \times \mathbf{Z}_2$-holonomy of the type $(CY \times S^1)/g.\mathcal{I}$, where g is a suitably defined freely-acting antiholomorphic involution on the CY which is precisely the same as the one considered in [23], Ω is the world-sheet parity and $\mathcal{I}$ reflects the S^1. These 7-manifolds are referred to as "barely G_2 manifolds" in [13]. In addition, the $D = 4$, $\mathcal{N} = 1$ Heterotic/F-theory dual models constructed have the following in common (as a consequence of applying fiberwise duality to Heterotic on T^2 being dual to F-theory on $K3$). The Heterotic theory is compactified on a CY_3 that is elliptically fibered over a 2-manifold B_2. The F-theory dual of this Heterotic theory is constructed by considering an elliptically fibered Calabi-Yau 4-fold X_4 that is elliptically fibered over a 3-manifold B_3. Additionally, the base B_3 is a $\mathbf{P}^1$-fibration over B_2 (the same one that figures on the heterotic side). We propose that the required Calabi-Yau 4-fold on the F-theory side is elliptically fibered over a trivially rationally ruled base given by $\mathbf{CP}^1 \times \mathcal{E}$, $\mathcal{E}$ being the Enriques surface. We raise an apparent puzzle regarding the derived Hodge data and the one that one might have naively guessed based on string/M/F dualities.

We now construct the M-theory uplift of type IIA background of [23]. Now, the specific $\mathcal{N}=1$ Heterotic/type IIA dual pair of [23] that we will be considering in this letter is Heterotic on a CY given by $\frac{K3\times T^2}{\mathbf{Z}_2^E}$ and type IIA on orientifolds of CY's (the mirrors of which are) given as hypersurface of degree 24 in $\mathbf{WCP}^4[1,1,2,8,12]$, the mirror duals to which, are given by:

$$z_1^{24}+z_2^{24}+z_3^{12}+z_4^3+z_5^2-12\alpha z_1z_2z_3z_4z_5-2\beta z_1^6z_2^6z_3^6-\gamma z_1^{12}z_2^{12}=0. \tag{44}$$

The $\mathbf{Z}_2^E$ represents the Enriques involution times reflection of the T^2 as considered. and the space-time orientation reversing antiholomorphic involution used for constructing the CY orientifold is:

$$\omega:(z_1,z_2,z_3,z_4,z_5)\rightarrow(\bar{z}_2,-\bar{z}_1,\bar{z}_3,\bar{z}_4,\bar{z}_5). \tag{45}$$

Another point worth keeping in mind is that under ω of (45), the Kähler form J going over to -J is only a statement in the cohomology group $H^{1,1}$. One can define inhomogenous coordinates for, e.g., Y, in the $z_2\neq 0$ coordinate patch:

$$u\equiv\frac{z_1}{z_2};\ v\equiv\frac{z_3}{z_2^2};\ w\equiv\frac{z_4}{z_2^4}, \tag{46}$$

[using which one can solve for $\frac{z_5}{z_2^{12}}$ from the defining equation (44), and hence is not included as part of the CY coordinate system]. Then, one can show that

$$J\stackrel{\omega}{\rightarrow}-J+\mathcal{O}\biggl(\frac{1}{|u|^{m>0}}\text{ or } g_{u\bar{u}}-\text{independent terms}\biggr), \tag{47}$$

such that the $-J$ and $-J+$ extra terms both belong to the same cohomology class of $H^{1,1}$. As $u\in\mathbf{CP}^1$-base coordinate and $g_{u\bar{u}}$ gives the size of the $\mathbf{CP}^1$ base, in the large base-limit of [23], J under the antiholomorphic involution ω goes over to $-J$ **exactly**. Similarly, $H^{2,1}$ goes over to $H^{1,2}$ (and $X_{2,1}\in H^{2,1}$ goes over to $X_{1,2}\in H^{1,2}$ exactly in the large-base limit of [23]) but an element $Y^{1,1}$ of $H^{1,1}$ goes over to an element of the cohomology class $[-Y^{1,1}]$ of $H^{1,1}$ and no statement can be made for large base-limit exactness like the ones for J or Ω above. The exact expressions for J and an element of $H^{2,1}$ under the action of ω is given in [7]. To summarize, we get:

$$\begin{aligned}&[\omega^*(J)]=[-J];\ \omega^*(J)\stackrel{\text{large }\mathbf{CP}^1}{\longrightarrow}-J,\\&[\omega^*(X)]=[\bar{X}];\ \omega^*(X)\stackrel{\text{large }\mathbf{CP}^1}{\longrightarrow}\bar{X};\\&[\omega^*(Y)]=[-Y],\end{aligned} \tag{48}$$

where $X\in H^{2,1}(CY_3\longrightarrow_{K3}\mathbf{CP}^1)$ and $Y\in H^{1,1}(CY_3\longrightarrow_{K3}\mathbf{CP}^1)$, and [ ] denotes the cohomology class. The closed and co-closed calibration 3-form ϕ is given by:

$$\phi=J\wedge dx+Re(e^{-\frac{i\theta}{2}}\Omega), \tag{49}$$

where x is the S^1 coordinate, and Ω is the holomorphic 3-form of the $CY_3(3,243)$. To get the spectrum for M-theory compactified on the 'barely G_2 manifold' $\mathcal{Z}\equiv\frac{CY\times S^1}{\omega.\mathcal{I}}$, one sees (See [13]) that

$\frac{1}{2}(H^{3,0}(CY) + H^{0,3}(CY))$ corresponding to $\frac{1}{2}(h^{3,0}(CY) + h^{0,3}(CY)) = 1$, is invariant under the $\mathbf{Z}_2$ involution ω. Similarly, $\frac{1}{2}(H^{2,1}(CY) + H^{1,2}(CY))$ corresponding to $\frac{1}{2}(h^{1,2}(CY) + h^{2,1}(CY)) = h^{2,1}(CY)$ elements, is invariant under the involution ω. As shown in [23], ω acts as -1 on $H^{1,1}(CY)$ implying that $H^{1,1}_+(CY)$, i.e., the part of $H^{1,1}(CY)$ even under ω is zero, and the part odd, $H^{1,1}_-(CY) = H^{1,1}(CY)$. Hence, n_V, n_C that denote respectively the number of vector and hypermultiplets, will be given by:

$$
\begin{aligned}
& n_V(\mathcal{Z}) = h^{1,1}_+(CY) = 0, \\
& n_C(\mathcal{Z}) = h^{2,1}(CY) + h^{3,0}(CY) + h^{1,1}_-(CY) = 243 + 1 + 3 = 247.
\end{aligned} \tag{50}
$$

This one sees that the spectra associated with Heterotic on $\frac{K3\times T^2}{\mathbf{Z}_2}$, type IIA on $\frac{CY}{\omega.\Omega}$, and M-theory on $\frac{CY\times S^1}{\omega.\mathcal{I}}$ match.

We now show the possibility of finding an $\mathcal{N}=1$ triality between the $\mathcal{N}=1$ heterotic on $CY_3(11,11)$(/type IIA on $\frac{CY_3(3,243)}{\omega.\Omega}$ dual pair) of Vafa-Witten, M theory on the "barely G_2 manifold" $\frac{CY_3(3,243)\times S^1}{\omega.\mathcal{I}}$ of $SU(3)\times\mathbf{Z}_2$ holonomy, and F-theory on an elliptically fibered X_4 , where the "11,11" and "3,243" denote the Hodge numbers, ω is an orienation-reversing antiholomorphic involution, $\mathcal{I}$ reverses the S^1. The X_4 that we obtain in this section will be obtained by assuming that the required F-theory dual must exist. Of course, given the basic string/M/F dualities, we know that such an F-theory dual must exist - what we show is given the same, what the geometric data of the required X_4 must be.

The CY_3 on the heterotic side that we are interested in is one that is obtained by a freely-acting Enriques involution acting on the $K3$ times a reflection of the T^2, in $K3\times T^2$, i.e., the Voisin-Borcea elliptically fibered $CY_3(11,11) \equiv \frac{K3\times T^2}{g.\mathcal{I}}$, where g is the generator of the Enriques involution and $\mathcal{I}$ reflects the T^2. Hence, the B_2 above is $\frac{K3}{g}$. Now, the $\mathcal{N}=2$ dual pair in [26] consisted of embedding $SU(2)\times SU(2)$ in $E_8\times E_8$ on the Heterotic side, resulting in $E_7\times E_7$, which is then Higgsed away. All that survives from the T^2 in $K3\times T^2$ are the abelian gauge fields corresponding to $U(1)^4$. As shown in Vafa-Witten's paper[23], in the $\mathcal{N}=1$ dual pair obtained by suitable $\mathbf{Z}_2$-moddings of both sides of the $\mathcal{N}=2$ Heterotic/type IIA dual pair, the $U(1)^4$ gets projected out so that there are no vector multiplets and one gets 247 $\mathcal{N}=1$ chiral multiplets on the Heterotic side on $CY_3(11,11)$. We should be able to get the same spectrum on the F-theory side. If r denotes the rank of the unbroken gauge group in Heterotic theory, then the number of $\mathcal{N}=1$ chiral multiplets in F-theory is given by the formula ([27, 28]):

$$
n_C = \frac{\chi(X_4)}{6} - 10 + h^{2,1}(X_4) - r, \tag{51}
$$

which excludes the S modulus of the Heterotic theory. The rank r in turn is expressed as:

$$
r = h^{1,1}(X_4) - h^{1,1}(B_3) - 1 + h^{2,1}(B_3). \tag{52}
$$

For Heterotic theory on $CY_3(11,11)$, $r=0$.

The fibration structure can be summarized as: $X_4 \longrightarrow_{T^2} B_3 \longrightarrow_{\mathbf{CP}^1} B_2 \equiv \frac{K3}{g} \equiv \mathcal{E} \equiv$Enriques surface. Given that for elliptically fibered X_4, $h^{1,1}(X_4) - h^{1,1}(B_3) - 1 \geq 0$, $r=0$ implies that

$$
\begin{aligned}
& h^{1,1}(X_4) = h^{1,1}(B_3) + 1 > 0; \\
& h^{2,1}(B_3) = 0.
\end{aligned} \tag{53}
$$

We can write the total number of Heterotic moduli

$$N_{het} = h^{1,1}(Z) + h^{2,1}(Z) + n_{bundle}, \tag{54}$$

where the bundle moduli correspond to an involution τ which acts trivially on the base and as reflection of the fiber (that can always be defined on an elliptically fibered Z [29]). It no longer can be defined as $h^1(Z, ad(V)) = I + 2n_o$, where the character-valued index I is given by -$\sum_{i=}^{3}(-)^i Tr_{H^i(Z,Ad(V))}(\frac{1+\tau}{2}) = -\sum_{i=0}^{3}(-)^i h_e^i(Z, Ad(V)) = n_e - n_o$ for no unbroken $\mathcal{N} = 1$ gauge group, and e, o referring to even,odd respectively under the involution τ. However, given that such an involution τ exists, one can still write that

$$n_{bundle} = n_e + n_o = \mathcal{I} + 2n_o, \tag{55}$$

for a suitable "index" $\mathcal{I}$. We assume that at the τ-invariant point, the action of τ can be lifted to an action of the gauge bundle embedded at the level of $K3$. This index will encode the information about $I(K3, Ad(SU(2) \times SU(2)))$ and the Higgsing away of the $E_7 \times E_7$, or equivalently $I(K3, Ad(E_8 \times E_8))$ at the $\mathcal{N} = 2$ level, and the freely acting Enriques involution times reflection of T^2. In general, one can always write the index $\mathcal{I}$ as $a + b\int_{\mathcal{E}} c_1^2(\mathcal{E}) + c\int_{\mathcal{E}} c_2(\mathcal{E}) + d\int_{\mathcal{E}} c_1^2(\mathcal{T}) + e\int_{\mathcal{E}} c_2(\mathcal{T}) + f\int_{\mathcal{E}} c_1(\mathcal{E}) \wedge c_1(\mathcal{T})$, where a, b, c, d, e, f are constants and $\mathcal{T}$ is a line bundle over $\mathcal{E}$. There are no non-perturbative Heterotic 5-branes in the $\mathcal{N} = 1$ model of [23]. Hence, for the $\mathcal{N} = 1$ Heterotic/F-theory duality to hold, there will no F-theory 3-branes(given by $\frac{\chi(X_4)}{24}$) either, which implies that the elliptically fibered Calabi-Yau 4-fold must satisfy the constraint:

$$\chi(X_4) = 0. \tag{56}$$

Assuming only a single section of the elliptic fibration: $Z \to_{T^2} \mathcal{E}(\equiv$ Enriques surface) and no 4-flux, from general considerations (See [30]), the Hodge data of X_4 will be given by:

$$\begin{aligned}
& h^{1,1}(X_4) = h^{1,1}(Z) + 1 + r = 12 - \int_{\mathcal{E}} c_1^2(\mathcal{E}) + r, \\
& h^{2,1}(X_4) = n_o, \\
& h^{3,1}(X_4) = h^{2,1} + \mathcal{I} + n_o + 1 = 12 + 29\int_{\mathcal{E}} c_1^2(\mathcal{E}) + \mathcal{I} + h^{2,1}(X_4).
\end{aligned} \tag{57}$$

Now $t \equiv c_1(\mathcal{T})$ ($\mathcal{T}$ being a line bundle over B_2), the analog of n in the Hirzebruch surface F_n, is a measure of the non-triviality of the $\mathbf{CP}^1$-fibration of the rationally ruled B_3. Now, the $CY_3(3, 243)$ on the type IIA side, can be represented as elliptic fibration over the Hirzebruch surface F_n, where n denotes the non-triviality of fibration of $\mathbf{CP}^1_f$ over $\mathbf{CP}^1_b$. The Weierstrass equation for $n = 0$ is given by:

$$y^2 = x^3 + \sum_{i=0}^{8} f_i^{(8)}(z_1) z_2^i x + \sum_{i=0}^{12} g_i^{(12)}(z_1) z_2^i, \tag{58}$$

implying that the number of complex structure deformations, $h^{2,1}$ is given by $9\times9+13\times13-(3+3+1) = 243$. [2] Hence, analogous to setting $n = 0$, we can set $t = 0$ and doing so would imply the triviality of the fibration: $B_3 = \mathbf{CP}^1 \times B_2 = \mathbf{CP}^1 \times \mathcal{E}$, for which $h^{2,1}(B_3) = 0$ thereby satisfying (53).

[2] Interestingly, for $n = 2$, the Weierstrass equation is given by:

$$y^2 = x^3 + \sum_{i=-4}^{4} f_{8-4i}(z_1) z_2^{4-i} x + \sum_{i=-8}^{8} g_{12-2i}(z_1) z_2^{8-i}, \tag{59}$$

Equating n_{het} to 246, one gets from (54) and (55) the following $\mathcal{I} + 2n_o = 224$. There are no vector multiplets, and in addition to the heterotic dilaton, n_{het} has to correspond to the number of $\mathcal{N} = 1$ chiral multiplets n_C on the F-theory side. Given that $r = \chi(X_4) = 0$, from (51) one gets: $h^{2,1}(X_4) = 128 = n_o$. This gives $\mathcal{I} = -32$. Using the relation: $\frac{\chi(X_4)}{6} = 8 + h^{1,1}(X_4) - h^{2,1}(X_4) + h^{3,1}(X_4)$, one sees that the elliptically 4-manifold X_4 that we are looking for is characterized by:

$$h^{1,1}(X_4) = 12, h^{2,1}(X_4) = 128, h^{3,1}(X_4) = 108. \tag{60}$$

This is consistent with (57). The $h^{2,2}(X_4)$ can be determined by the following relation [31] $h^{2,2}(X_4) = 2(22 + 2h^{1,1}(X_4) + 2h^{3,1}(X_4) - h^{2,1}(X_4)) = 268$, which has been obtained from the definitions of elliptic genera in terms of hodge numbers and as integrals involving suitable powers of suitable Chern classes, and $c_1(X_4) = 0$. Hence, $\mathcal{N} = 1$ Heterotic Theory on $\frac{K3\times T^2}{\mathbf{Z_2}}$ is dual to F-theory on an elliptically fibered Calabi-Yau 4-fold: $X_4[h^{1,1} = 12, h^{2,1} = 128, h^{3,1} = 108; 0] \rightarrow_{T^2} \mathbf{CP}^1 \times \mathcal{E}$. We now discuss an apparent puzzle. At the $\mathcal{N} = 2$ level, Heterotic on $K3 \times T^2$ should be dual to F-theory on $CY_3(3, 243) \times T^2$ as a consequence of repeated fiberwise application of duality to the basic duality that Heterotic on T^2 is dual to F-theory on $K3$, as well as because type IIA on a CY_3 should be dual to F-theory on $CY_3 \times T^2$ and Heterotic on $K3 \times T^2$ is dual to type IIA on $CY_3(3, 243)$. Hence, it is possible that an orbifold of $K3 \times T^2$ on the Heterotic side should correspond to a suitable orbifold of $CY_3 \times T^2$ on the F-theory side. Note, however, even though a naive freely acting orbifold of $CY_3(3, 243) \times T^2$ gives the right null Euler Characteristic, it can not, for instance, give $h^{1,1} = 12$, i.e., an enhancement over the $h^{1,1}(CY_3(3, 243) \times T^2) = 3 + 1 = 4$. This is unlike the case of the F-theory dual of Heterotic on Voisin-Borcea $CY_3(19, 19)$ which corresponded to an involution with fixed points, considered in [28]. Of course, given the string/M/F dualities, the X_4 with the derived fibration structure and Hodge data must exist, as the F-theory dual corresponding to Heterotic on $CY_3(11, 11)$ must exist. One needs to look further into this issue.

The CY_4 with the required fibration structure and Hodge data given in as derived above is missing from the list of hypersurfaces in $\mathbf{WCP}^5$ of Kreuzer and Skarke because it is not possible to get the desired CY_4 as a hypersurface in any toric variety as fibrations of toric hypersurfaces have bases that are toric varieties, and the Enriques surface, $\mathcal{E}$, is not a toric variety. Perhaps, one needs a "nef partition" (one could use "nef.x" part of the package PALP[32]) that makes the base, $\mathbf{CP}^1 \times \mathcal{E}$ a toric hypersurface. One might have to work with complete intersections in toric varieties.

3.2 Type IIA on $CY_3(3, 243)$ and $D = 11$ Supergravity Uplift of its Orientifold

The periods are the building blocks, e.g., for getting the prepotential in $\mathcal{N} = 2$ type II theories compactified on a Calabi-Yau. It is in this regard that the Picard-Fuchs equation satisfied by the periods, become quite important. In [7], we addressed the issue of deriving the Picard-Fuchs equation on the mirror Landau-Ginsburg side corresponding to the gauged linear sigma model for a compact Calabi-Yau $CY_3(3, 243)$, expressed as a degree-24 Fermat hypersurface in a suitableweighted complex projective space, but staying away from the orbifold singularities by taking the large-base limit of the compact Calabi-Yau.

implying that the number of complex structure deformations, $h^{2,1}$ is given by (17+15+13+...+3+1 =)81+(25+23+...+3+1 =)169 − (3 + 3 + 1) = 243. Hence, elliptic fibrations over both F_0 and F_2 give the same hodge numbers. We will work with F_0.

Even though, one ended up with more than the required number of solutions, but the essential idea that was highlighted was the ease with which, both the large and small complex structure limits could be addressed, and the fact that the nonanalytic ln-terms in the periods, could be easily obtained without having to resort to parametric differentiations of infinite series. In this paper, we address the problem of getting the right number of the right kind of solutions on the mirror Landau-Ginsburg side, but this time after having resolved the orbifold singularities. We also address the problems of showing that unoriented instantons do not generate a superpotential on the type IIA side in the $\mathcal{N}=1$ Heterotic/type IIA dual pair of [23], whose M and F theory uplifts were discussed in [6]. It was shown in [7], using mirror symmetry, that as expected from the Heterotic and F theory duals, there is indeed no superpotential generated from $\mathbf{RP}^2$-instantons in the type IIA side in the large-base limit of $CY_3(3,243)$, away from the aforementioned orbifold singularities of the relevant Fermat hypersurface. In this paper, we show that the same remains true even after the resolution of the orbifold singularities. Further, we discuss the supergravity uplift of the type IIA orientifold that figures in the abovementioned $\mathcal{N}=1$ Heterotic/type IIA dual pair, to $D=11$ supergravity. We evaluate the Kähler potential in the large volume limit of $CY_3(3,243)$. As an interesting aside, we give a conjecture about the action of the antiholomorphic involution that figures in the definition of the type IIA orientifold, on the periods, given its action on the cohomology of $CY_3(3,243)$, using a canonical (co)homology basis to expand the holomorphic 3-form. We verify the conjecture for T^6 and (partly) for the mirror to the quintic.

By following the alternative formulation of Hori and Vafa [33] for deriving the Picard-Fuchs equation for a definition of period integral in the mirror Landau-Ginsburg model, we obtain solutions valid in the large *and* small complex structure limits, and get the ln terms as naturally as the analytic terms (i.e. without using parametric differentiation of infinite series). We also study in detail, the monodromy matrix in the large and small complex structure limits.

Consider the Calabi-Yau 3-fold given as a degree-24 Fermat hypersurface in the weighted projective space $\mathbf{WCP}^4[1,1,2,8,12]$:

$$P = z_1^{24} + z_2^{24} + z_3^{12} + z_4^3 + z_5^2 = 0. \tag{61}$$

It has a $\mathbf{Z}_2$-singularity curve and a $\mathbf{Z}_4$-singularity point. $\mathbf{Z}_2$ and $\mathbf{Z}_4$ singularity resolution $\leftrightarrow$ The two new chiral superfields needed to be introduced as a consequence of singularity resolution, correspond to the two $\mathbf{CP}^1$'s that required to be introduced in blowing up the singularities. One then has to consider three instead of a single C^* action, and the $CY_3(3,243)$ [3] can be expressed as a suitable holomorphic quotient corresponding to a smooth toric variety. To be more specific, one considers the resolved Calabi-Yau $CY_3(3,243)$ as the holomorphic quotient: $\frac{C^7-F}{(C^*)^3}|_{\text{hyp constraint}}$, where the diagonal $(C^*)^3$ actions on the seven coordinates of C^7 are given by:

$$x^j \sim \lambda^{iQ_j^a} x^j, \text{ no sum over } j;\ a = 1,2,3, \tag{62}$$

where the three sets of charges $\{Q_{i=(0,),1,\ldots,7}^{a=1,2,3}\}$ (the "0" being for the extra chiral superfield with $Q_i^0 =$

[3] The $CY_3(3,243)$ considered in this paper will be an elliptic fibration over the Hirzebruch surface F_2.

$-\sum_{i=1}^7 Q_i^a[34]$) are given by the following:

$$\begin{array}{c|cccccccc} & \mathcal{X}_0 & \mathcal{X}_1 & \mathcal{X}_2 & \mathcal{X}_3 & \mathcal{X}_4 & \mathcal{X}_5 & \mathcal{X}_6 & \mathcal{X}_7 \\ \hline Q_i^{(1)}: & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ Q_i^{(2)}: & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ Q_i^{(3)}: & -6 & 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{array} \tag{63}$$

where on noting:

$$Q^{(1)} + 2Q^{(2)} + 4Q^{(3)} = \quad -24 \quad 1 \quad 1 \quad 0 \quad 2 \quad 8 \quad 12 \quad 0 \ , \tag{64}$$

one identifies $\mathcal{X}_{3,7}$ as the two extra chiral superfields introduced as a consequence of singularity resolution.

The Landau-Ginsburg Period for the resolved $CY_3(3,243)$, as per the prescription of Hori and Vafa, is given by:

$$\begin{aligned} \Pi(t_1,t_2,t_3) &= \int \prod_{i=0}^7 dY_i \prod_{a=1}^3 dF^{(a)} \sum_{a=1}^3 d_{1a} F^{(a)} e^{-\sum_{a=1}^3 F^{(a)}(\sum_{i=1}^7 Q_i^{(a)} Y_i - Q_0^{(a)} Y_0 - t_{(a)}) - \sum_{i=0}^7 e^{-Y_i}} \\ &= \sum_{a=1}^3 d_{1a} \frac{\partial}{\partial t_{(a)}} \int \prod_{i=0}^7 dY_i \prod_{a=1}^3 \delta(\sum_{i=1}^7 Q_i^{(a)} Y_i - Q_0^{(a)} Y_0 - t_{(a)}) e^{-\sum_{i=0}^7 e^{-Y_i}} \\ &\equiv \sum_{a=1}^3 d_{1a} \frac{\partial}{\partial t_{(a)}} \tilde{\Pi}(t_{1,2,3}); \end{aligned} \tag{65}$$

$d_{\alpha a} \equiv$ charge matrix, α indexes the number of hypersurfaces and a indexes the number of $U(1)$'s. For $CY_3(3,243)$, $\alpha = 1$, $a = 1,2,3$ with $d_{11} = d_{12} = 0, d_{13} = 6$.

Consider:

$$\tilde{\Pi}(t_{1,2,3}, \{\mu_i\}) \equiv \int \prod_{i=0}^7 dY_i \prod_{a=1}^3 \delta(\sum_{i=1}^7 Q_i^{(a)} Y_i - Q_0^{(a)} Y_0 - t_{(a)}) e^{-\sum_{i=0}^7 \mu_i e^{-Y_i}}. \tag{66}$$

One can show that:

$$\tilde{\Pi}(t_{1,2,3}, \{\mu_i\}) = \tilde{\Pi}(t'_{1,2,3}, \{\mu_i = 1\}), \tag{67}$$

where

$$t'_1 \equiv t_1 + ln(\mu_3^2/\mu_1\mu_2), \ t'_2 = t_2 + ln(\mu_7^2/\mu_3\mu_4), \ t'_3 = t_3 + ln(\mu_0^6/\mu_7\mu_5^2\mu_6^3). \tag{68}$$

Eliminating $Y_{0,3,7}$ gives a order-24 Picard-Fuchs equation:

$$\frac{\partial^{24}}{\partial\mu_1\partial\mu_2\partial\mu_4^2\partial\mu_5^8\partial\mu_6^{12}} \tilde{\Pi}(t_{1,3,4}) = e^{-t_1+2t_2+4t_3} \frac{\partial^{24}}{\partial\mu_0^{24}} \tilde{\Pi}(t_{1,2,3}), \tag{69}$$

which is the same as the PF equation for the unresolved hypersurface away from the orbifold singularities. **This overcounts the number of solutions**.

The right number of solutions must be $2h^{2,1}(\text{Mirror}) + 2 = 2.3 + 2 = 8$. To get this number, one notes that by adding the three constraints:

$$Y_1 + Y_2 - 2Y_3 = t_1; \ Y_3 + Y_4 - 2Y_7 = t_2; \ -6Y_0 + 2Y_5 + 3Y_6 + Y_7 = t_3, \tag{70}$$

one gets:

$$-6Y_0 - Y_3 - Y_7 + Y_1 + Y_2 + Y_4 + 2Y_5 + 3Y_6 = t_1 + t_2 + t_3, \tag{71}$$

which allows one to write the following order-8 PF equation:

$$\frac{\partial^8}{\partial\mu_1\partial\mu_2\partial\mu_4\partial\mu_5^2\partial\mu_6^3}\tilde{\Pi}(t_{1,2,3}) = e^{-(t_1+t_2+t_3)}\frac{\partial^8}{\partial\mu_0^6\partial\mu_3\partial\mu_7}\tilde{\Pi}(t_{1,2,3}). \tag{72}$$

If $\Theta_i \equiv \frac{\partial}{\partial t'_i}$, then one gets:

$$\left[\Theta_1^2\Theta_2\prod_{l=2}^{3}\prod_{k=0}^{l-1}(-l\Theta_3 - k) - e^{-(t'_1+t'_2+t'_3)}(2\Theta_2 - \Theta_3)(2\Theta_1 - \Theta_2)\prod_{j=0}^{5}(6\Theta_3 - j)\right]\tilde{\Pi} = 0$$

with $z \equiv e^{-(t'_1+t'_2+t'_3)}$; $z\dfrac{d}{dz} \equiv \Delta_z$, and rescaling :

$$\left[\Delta_z^4(\Delta_z - \frac{1}{2})\Delta_z(\Delta_z - \frac{1}{3})(\Delta_z - \frac{2}{3}) + z\prod_{j=0}^{5}(\Delta_z + \frac{j}{6})\Delta_z^2\right]\tilde{\Pi} = 0. \tag{73}$$

One solution to the above equation is: ${}_8F_7\left(\begin{array}{cccccccc} 0 & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \frac{5}{6} & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{2} & 1 & \frac{2}{3} & \frac{1}{3} & \end{array}\right)(-z)$

$\Big[$ For $e^{-t'} \equiv z$, $t' \equiv t'_1 + 2t'_2 + 4t'_3$ and suitable rescaling of z, the relevant order-24 PF equation **for the unresolved hypersurface** is:

$$\Delta_z^2\Delta_z(\Delta_z - \frac{1}{2})\prod_{j=1}^{8}(\Delta_z - \frac{j-1}{8})\prod_{j=1}^{12}(\Delta_z - \frac{j-1}{12})\tilde{\Pi} = z\prod_{j=1}^{24}(\Delta_z + \frac{j-1}{24})\tilde{\Pi}. \tag{74}$$

One solution can be written in terms of the following generalized hypergeometric function

$${}_{24}F_{23}\left(\begin{array}{cccccccc} 0 & \frac{1}{24} & \frac{2}{24} & \frac{3}{24} & \frac{4}{24} & \frac{5}{24} & .... & \frac{23}{24} \\ 1 & 1 & \frac{1}{2} & \frac{5}{8} & ... - \frac{2}{8} & \frac{1}{12} & ... & -\frac{10}{12} \end{array}\right). \tag{75}$$

$\Big]$

From the above solution, Meijer basis obtained using properties of ${}_pF_q$ and the Meijer function I:

$${}_pF_q\left(\begin{array}{ccccc} \alpha_1 & \alpha_2 & \alpha_3 & .... & \alpha_p \\ \beta_1 & \beta_2 & \beta_3 & .... & \beta_q \end{array}\right)(z) = \frac{\prod_{i=1}^{p}\Gamma(\beta_i)}{\prod_{j=1}^{q}\Gamma(\alpha_j)}I\left(\begin{array}{c|c} 0 & \alpha_1...\alpha_p \\ \hline . & \beta_1...\beta_q \end{array}\right)(-z) \text{ where}$$

$$I\left(\begin{array}{c|c} a_1...a_A & b_1...b_B \\ \hline c_1...c_C & d_1...d_D \end{array}\right)(z), I\left(\begin{array}{c|c} a_1...(1-d_l)...a_A & b_1...b_B \\ \hline c_1...c_C & d_1...\hat{d}_l...d_D \end{array}\right)(-z)$$

$$I\left(\begin{array}{c|c} a_1...a_A & b_1..\hat{b}_j...b_B \\ \hline c_1..(1-b_j)..c_C & d_1...d_D \end{array}\right)(-z) \tag{76}$$

satisfy the same equation.

Now, $z \equiv e^{-(t_1'+t_2'+t_3')} = e^{-(t_1+t_2+t_3)}\frac{\mu_1\mu_2\mu_4\mu_5^2\mu_6^3}{\mu_0^6\mu_3\mu_7}$. Hence, one can solve for large ($\equiv |z| << 1$) and small complex structure ($\equiv |z| >> 1$) limits, as well as large-size-Calabi-Yau limit($\equiv t_i \rightarrow \infty \Leftrightarrow |z| << 1$) on the mirror Landau-Ginsburg side with equal ease using Mellin-Barnes integral represention for the Meijer's function I, as in [35] and in (77) below.

Now, to get an infinite series expansion in z for $|z| < 1$ as well as $|z| > 1$, one uses the following

$$I\left(\begin{array}{c|c} a_1...a_A & b_1...b_B \\ \hline c_1...c_C & d_1...d_D \end{array}\right)(z) = \frac{1}{2\pi i}\int_\gamma ds \frac{\prod_{i=1}^A \Gamma(a_i - s)\prod_{j=1}^B \Gamma(b_j + s)}{\prod_{k=1}^C \Gamma(c_k - s)\prod_{l=1}^D \Gamma(d_l + s)} z^s, \tag{77}$$

where the contour γ lies to the right of:$s+b_j = -m \in \mathbf{Z}^- \cup \{0\}$ and to the left of: $a_i - s = -m \in \mathbf{Z}^- \cup \{0\}$.

This, $|z| << 1$ and $|z| >> 1$ can be dealt with equal ease by suitable deformations of the contour γ (see Fig. 1) to γ' and γ'' respectively (See Fig. 2). Additionally, instead of performing parametric differentiation of infinite series to get the ln-terms, one get the same (for the large complex structure limit: $|z| < 1$) by evaluation of the residue at $s = 0$ in the Mellin-Barnes contour integral in (77) as is done explicitly to evaluate the eight integrals in (78).

The guiding principle is that of the eight solutions to $\tilde{\Pi}$, one should generate solutions in which one gets $(lnz)^P$, $P = 1, ..., 4$ so that one gets $(lnz)^{P-1}$ for Π, and one can then identify terms independent of lnz with Z^0, three (lnz) terms with $Z^{1,2,3}$, three $(lnz)^{P\leq 2}$ terms with $F_{1,2,3} \equiv \frac{\partial F}{\partial Z^{1,2,3}}$, and finally $(lnz)^{P\leq 3}$ term with $F_0 \equiv \frac{\partial F}{\partial Z^0}$.

One (non-unique) choice of solutions for $\tilde{\Pi}(z)$ is given below:

$$
z\frac{d}{dz}\left[\begin{array}{c}
I\left(\begin{array}{c|c} 0\ 0 & 0\ \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0\ 0 \\ \hline . & 1\ 1\ \frac{1}{2}\ 1\ \frac{2}{3}\ \frac{1}{3} \end{array}\right)(-z) \\
I\left(\begin{array}{c|c} 0\ 0\frac{1}{2}\ \frac{1}{3}\ \frac{2}{3} & \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6} \\ \hline 1\ 1\ 1 & 1\ 1\ 1 \end{array}\right)(-z) \\
I\left(\begin{array}{c|c} 0\ 0\ \frac{1}{2} & 0\ \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0 \\ \hline 1 & 1\ 1\ 1\ \frac{2}{3}\ \frac{1}{3} \end{array}\right)(-z) \\
I\left(\begin{array}{c|c} 0\ 0\ \frac{1}{3} & 0\ \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0 \\ \hline 1 & 1\ 1\ \frac{1}{2}\ 1\ \frac{1}{3} \end{array}\right)(z) \\
I\left(\begin{array}{c|c} 0\ 0\ \frac{2}{3} & 0\ \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0 \\ \hline 1 & 1\ 1\ \frac{1}{2}\ 1\ \frac{2}{3} \end{array}\right)(-z) \\
I\left(\begin{array}{c|c} 0\ 0\ \frac{1}{2} & \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0 \\ \hline 1\ 1 & 1\ 1\ 1\ \frac{2}{3}\ \frac{1}{3} \end{array}\right)(z) \\
I\left(\begin{array}{c|c} 0\ 0\ \frac{1}{3} & \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0 \\ \hline 1\ 1 & 1\ 1\ \frac{1}{2}\ \frac{2}{3}\ \frac{1}{3} \end{array}\right)(z) \\
I\left(\begin{array}{c|c} 0\ 0\ \frac{2}{3} & \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0 \\ \hline 1\ 1 & 1\ 1\ \frac{1}{2}\ \frac{2}{3} \end{array}\right)(z)
\end{array}\right] \sim \begin{bmatrix} F_0 \\ Z^0 \\ F_1 \\ F_2 \\ F_3 \\ Z^1 \\ Z^2 \\ Z^3 \end{bmatrix} \tag{78}
$$

(a)

$$I\left(\begin{array}{c|c} 0\ 0 & 0\ \frac{1}{6}\ \frac{2}{6}\ \frac{3}{6}\ \frac{4}{6}\ \frac{5}{6}\ 0\ 0 \\ \hline & \\ . & 1\ 1\ \frac{1}{2}\ 1\ \frac{2}{3}\ \frac{1}{3} \end{array}\right)(-z) = \frac{1}{2\pi i}\int_\gamma ds \frac{[\Gamma(-s)]^2[\Gamma(s)]^3\prod_{j=1}^5\Gamma(s+\frac{j}{6})}{[\Gamma(s+1)]^3\Gamma(\frac{1}{2}+s)\Gamma(\frac{2}{3}+s)\Gamma(\frac{1}{3}+s)}(-z)^s$$

$$= \theta(1-|z|)\Biggl[\frac{2(2\pi)^{\frac{3}{2}}}{\sqrt{\pi}}\Biggl[\left(ln\left(\frac{2^2 3^3}{6^6}\right)+ln(-z)\right)^4+\frac{65}{3}\left(ln\left(\frac{2^2 3^3}{6^6}\right)+ln(-z)\right)^2$$

$$+\frac{169\pi^4}{3} - 1440\zeta(3)\left(ln\left(\frac{2^2 3^3}{6^6}\right)+ln(-z)\right)+402\zeta(4)\Biggr]+\sum_{m=1}^\infty \frac{(2\pi)^{\frac{3}{2}}2^{\frac{1}{2}-6m}\Gamma(6m)}{m^3(m!)^2 3^{3m}\Gamma(\frac{1}{2}+m)\Gamma(3m)}$$

$$\times\left[2\gamma+6\Psi(6m)-2\Psi(2m)-3\Psi(m)-3m^2+ln\left(\frac{2^2 3^3}{6^6}\right)\right](-z)^m\Biggr]$$

$$-\theta(|z|-1)\Biggl[\sum_{m=0}^\infty \frac{(-)^m(\Gamma(m+\frac{1}{6})^2\Gamma(-m+\frac{2}{3})}{(m+\frac{1}{6})^3 m!}(-z)^{-m-\frac{1}{6}}+$$

$$\sum_{m=0}^\infty \frac{(-)^m(\Gamma(m+\frac{5}{6})^2\Gamma(-m-\frac{2}{3})}{(m+\frac{5}{6})^3 m!}(-z)^{-m-\frac{5}{6}}\Biggr] \tag{79}$$

The connection between (78) that *effectively depends only on one complex structure parameter* $z = e^{-(t_1+t_2+t_3)}\frac{1}{z_1 z_2 z_3}$, and the solutions given in the literature [36]of the form:

$$\partial^{s_m}_{\rho_m}\partial^{s_n}_{\rho_n}\partial^{s_p}_{\rho_p}\sum_{m,n,p} c(m,n,p;\rho_m,\rho_n,\rho_p)z_1^{m+\rho_m}z_2^{n+\rho_n}z_3^{p+\rho_p}|_{\rho_m=\rho_n=\rho_p=0}, \tag{80}$$

with $s_m+s_n+s_p\le 3$, and $z_1\equiv\frac{\mu_1\mu_2}{\mu_3^2},\ z_2\equiv\frac{\mu_3\mu_4}{\mu_7^2},\ z_3\equiv\frac{\mu_7\mu_5^2\mu_6^3}{\mu_0^6}$, needs to be understood. The appearance of $\partial^{s_m}_{\rho_m}\partial^{s_n}_{\rho_n}\partial^{s_p}_{\rho_p}\sum_{m,n,p}$ in (80) is what was referred to earlier on as parametric differentiation of infinite series, something which, as we have explicitly shown above, is not needed in the approach followed in this work.

The Picard-Fuchs equation can be written in the form[37]:

$$\left(\Delta_z^8+\sum_{i=1}^7\mathbf{B}_i(z)\Delta_z^i\right)\tilde{\Pi}(z)=0. \tag{81}$$

The Picard-Fuchs equation in the form written in (81) can alternatively be expressed as the following system of eight linear differential equations:

$$\Delta_z\left(\begin{array}{c}\tilde{\Pi}(z)\\ \Delta_z\tilde{\Pi}(z)\\ (\Delta_z)^2\tilde{\Pi}(z)\\ ...\\ (\Delta_z)^7\tilde{\Pi}(z)\end{array}\right)=$$

$$\begin{pmatrix} 0 & 1 & 0 & ...0 & 0 \\ 0 & 0 & 1 & ...0 & 0 \\ . & . & . & .... & . \\ 0 & 0 & 0 & ...0 & 1 \\ 0 & -\mathbf{B}_1(z) & -\mathbf{B}_2(z) & ... -\mathbf{B}_6(z) & -\mathbf{B}_7(z) \end{pmatrix} \begin{pmatrix} \tilde{\Pi}(z) \\ \Delta_z\tilde{\Pi}(z) \\ (\Delta_z)^2\tilde{\Pi}(z) \\ ... \\ (\Delta_z)^7\tilde{\Pi}(z) \end{pmatrix} \tag{82}$$

The matrix on the RHS of (82) is usually denoted by $A(z)$.

If the eight solutions, $\{\tilde{\Pi}_{I=1,...,8}\}$, are collected as a column vector $\tilde{\Pi}(z)$, then the *constant*[4] monodromy matrix T for $|z| << 1$ is defined by:

$$\tilde{\Pi}(e^{2\pi i}z) = T\tilde{\Pi}(z). \tag{83}$$

The basis for the space of solutions can be collected as the columns of the "fundamental matrix" $\Phi(z)$ given by:

$$\Phi(z) = S_8(z)z^{R_8}, \tag{84}$$

where $S_8(z)$ and R_8 are 8×8 matrices that single and multiple-valued respectively. Note that $\mathbf{B}_i(0) \neq 0$, which influences the monodromy properties. Also,

$$\Phi(z)_{ij} = \begin{pmatrix} \tilde{\Pi}_1(z) & ... & \tilde{\Pi}_8(z) \\ \Delta_z\tilde{\Pi}_1(z) & ... & \Delta\tilde{\Pi}_8(z) \\ \Delta_z^2\tilde{\Pi}_2(z) & ... & \Delta^2\tilde{\Pi}_8(z) \\ ... & ... & ... \\ \Delta_z^7\tilde{\Pi}_1(z) & ... & \Delta_z^7\tilde{\Pi}_8(z) \end{pmatrix}_{ij}, \tag{85}$$

implying that

$$T = e^{2\pi i R^t}. \tag{86}$$

Now, writing $z^R = e^{Rlnz} = 1 + Rlnz + R^2(lnz)^2 + ...$, and further noting that there are no terms of order higher than $(lnz)^4$ in $\tilde{\Pi}(z)$ obtained above, implies that the matrix R must satisfy the property: $R^m = 0,\ m = 5, ...\infty$. Hence, $T = e^{2\pi i R^t} = 1 + 2\pi i R^t + \frac{(2\pi i)^2}{2}(R^t)^2 + \frac{(2\pi i)^3}{6}(R^t)^3 + \frac{(2\pi i)^4}{24}(R^t)^4$. Irrespective of whether or not the distinct eigenvalues of $A(0)$ differ by integers, one has to evaluate $e^{2\pi i A(0)}$. The eigenvalues of $A(0)$ of (90), are $0^5, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and hence five of the eight eigenvalues differ by an integer (0).

Now, the Picard-Fuchs equation (73) can be rewritten in the form (81), with the following values of B_i's:

$$\mathbf{B}_{1,2} = 0, \mathbf{B}_3 = \frac{5z}{324(1+z)}, \mathbf{B}_4 = \frac{137}{648(1+z)}$$
$$\mathbf{B}_5 = \frac{(\frac{25}{24}z - \frac{1}{9})}{(1+z)}, \mathbf{B}_6 = \frac{(26+85z)}{36(1+z)}, \mathbf{B}_7 = -\frac{(3-5z)}{2(1+z)}. \tag{87}$$

[4] This thus implies that both $\tilde{\Pi}$ and Π, have the *same* monodromy matrix.

Under the change of basis $\tilde{\Pi}(z)\rightarrow\tilde{\Pi}'(z)=M^{-1}\tilde{\Pi}(z)$, and writing $\tilde{\Pi}_j(z)=\sum_{i=0}^4(lnz)^iq_{ij}(z)$ (See [35] and the appendix), one sees that

$$\begin{aligned}
&\tilde{\Pi}'_j(z)=\sum_{i=0}^4(lnz)^iq'_{ij}(z),\\
&q'(z)=q(z)(M^{-1})^t,\\
&\Phi'(z)\Phi(z)(M^{-1})^t,\ S'(z)=S(z)(M^{-1})^t,\ R'=M^tR(M^{-1})^t.
\end{aligned}\tag{88}$$

By choosing M such that $S'(0)=\mathbf{1}_{24}$, one gets

$$T(0)=M(e^{2i\pi A(0)})^tM^{-1}.\tag{89}$$

The matrix $A(0)$ is given by:

$$A(0)=\left(\begin{array}{cccccccc}
0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&\frac{1}{9}&-\frac{13}{18}&\frac{3}{2}
\end{array}\right)\tag{90}$$

Using MATHEMATICA, one then can evaluate the "matrix exponent" involving $A(0)$.
Writing the solution vector $\tilde{\Pi}_i$ as $\tilde{\Pi}_i=\sum_{j=0}^4(lnz)^jq_{ji}$ (following the notation of [35]), one notes:

$$(\Phi')_i=(\tilde{\Pi}')^t_i=\left(S'z^{A(0)}\right)_{0i}=(lnz)^jq'_{ji}.\tag{91}$$

From (91), one gets the following:

$$(q'(0))_{ji}=\frac{\delta_{ji}}{j!},\ 0\leq(i,j)\leq4.\tag{92}$$

For $5\leq i\leq7$, consider. e.g., $i=5$. Then from the expression for $z^{A(0)}$ above,

$$\begin{aligned}
&\sum_{j=0}^4(q')_{j5}(lnz)^j=(S')_{00}[f_{05}(z^{\frac{1}{2}},z^{\frac{1}{3}})+\sum_{n=1}^4c_n^{(05}(lnz)^n]+(S')_{01}[f_{15}(z^{\frac{1}{2}},z^{\frac{1}{3}})+\sum_{n=1}^3c_n^{(25)}(lnz)^n]\\
&(S')_{02}[f_{25}(z^{\frac{1}{2}},z^{\frac{1}{3}})+\sum_{n=1}^2c_n^{(25}(lnz)^n]+(S')_{03}[f_{35}(z^{\frac{1}{2}},z^{\frac{1}{3}})+c_1^{(35)}(lnz)]\\
&+(S')_{04}f_{45}(z^{\frac{1}{2}},z^{\frac{1}{3}})+(S')_{05}f_{55}(z^{\frac{1}{3}})+(S')_{06}f_{65}(z^{\frac{1}{2}},z^{\frac{1}{3}})+(S')_{07}f_{75}(z^{\frac{1}{3}})
\end{aligned}\tag{93}$$

where the f_{ij}'s and c_n^{ij}'s can be determined from the expression for $z^{A(0}$ given below. From (93), one gets:

$$(q')_{05} = \sum_{i=0}^{7}(S')_{0i}f_{i5}, (q')_{15} = \sum_{i=0}^{3}(S')_{0i}c_1^{i5}, (q')_{25} = \sum_{i=0}^{2}(S')_{0i}c_2^{i5}$$
$$(q')_{35} = \sum_{i=0}^{1}(S')_{0i}c_3^{i5}, (q')_{45} = (S')_{00}c_4^{05}. \tag{94}$$

From (94), one gets:

$$(q'(0))_{0i} = f_{0i}(0); (q'(0))_{ij} = c_i^{0j},\ 1 \leq i \leq 4,\ 5 \leq j \leq 7. \tag{95}$$

Again using the MATHEMATICA notebook format, the value of $z^{A(0)}$, as evaluated by MATHEMATICA is given by: The matrix q' introduced in (91) is given by:

$$q' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\left(\frac{391933}{32}\right) & \frac{130077}{32} & -\left(\frac{53739}{16}\right) \\ 0 & 1 & 0 & 0 & 0 & -\left(\frac{5971}{16}\right) & \frac{19215}{16} & -\left(\frac{7785}{8}\right) \\ 0 & 0 & \frac{1}{2} & 0 & 0 & -\left(\frac{865}{16}\right) & \frac{2637}{16} & -\left(\frac{1035}{8}\right) \\ 0 & 0 & 0 & \frac{1}{6} & 0 & -\left(\frac{115}{24}\right) & \frac{105}{8} & -\left(\frac{39}{4}\right) \\ 0 & 0 & 0 & 0 & \frac{1}{24} & -\left(\frac{13}{32}\right) & \frac{9}{16} & -\left(\frac{3}{8}\right) \end{pmatrix} \tag{96}$$

Further, the matrix q is of the form:

$$q = \begin{pmatrix} q_{00} & q_{01} & q_{02} & q_{03} & q_{04} & q_{05} & q_{06} & q_{07} \\ q_{10} & q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} & q_{17} \\ 0 & q_{21} & q_{22} & q_{23} & q_{24} & q_{25} & q_{26} & q_{27} \\ 0 & 0 & 0 & 0 & q_{34} & q_{35} & q_{36} & q_{37} \\ 0 & 0 & 0 & 0 & q_{44} & 0 & 0 & 0 \end{pmatrix} \tag{97}$$

From the matrix equation $q' = q(M^{-1})^t$, one sees that one has 40 equations in 64 variables, one has the freedom to (judiciously) give arbitrary values to 24 variables, bearing in mind that from the forms of the matrices q' and $q(M^{-1})^t$, the values of $(M^{-1})^t_{4i}$ are fixed. We set: $M_{ij} = \delta_{ij}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 7$.

The above matrix equation can then be solved for the 64-24=40 entries $(M^{-1})^t_{ij}$ $4 \leq i \leq 7,\ 0 \leq j \leq 7$ to give the following result:

$(M^{-1})^t_{30} = \frac{X_{30}}{Y_{30}}$, where

$X_{30} = q_{06}\, q_{10}\, q_{27}\, q_{35} + q_{16}\, q_{27}\, q_{35} - q_{00}\, q_{16}\, q_{27}\, q_{35} - q_{05}\, q_{10}\, q_{27}\, q_{36} - q_{15}\, q_{27}\, q_{36} + q_{00}\, q_{15}\, q_{27}\, q_{36}(-1 + q_{00})\, q_{17}\, (q_{26}\, q_{35} - q_{25}\, q_{36}) + q_{07}\, q_{10}\, (-(q_{26}\, q_{35}) + q_{25}\, q_{36}) -$
$q_{06}\, q_{10}\, q_{25}\, q_{37} - q_{16}\, q_{25}\, q_{37} + q_{00}\, q_{16}\, q_{25}\, q_{37} + q_{05}\, q_{10}\, q_{26}\, q_{37} + q_{15}\, q_{26}\, q_{37} - q_{00}\, q_{15}\, q_{26}\, q_{37}$

$Y_{30} = -(q_{03}\, q_{17}\, q_{26}\, q_{35}) + q_{03}\, q_{16}\, q_{27}\, q_{35} - q_{05}\, q_{17}\, q_{23}\, q_{36} + q_{03}\, q_{17}\, q_{25}\, q_{36} + q_{05}\, q_{13}\, q_{27}\, q_{36} - q_{03}\, q_{15}\, q_{27}\, q_{36} + q_{07}\, (-(q_{16}\, q_{23}\, q_{35}) + q_{13}\, q_{26}\, q_{35} + q_{15}\, q_{23}\, q_{36} - q_{13}\, q_{25}\, q_{36}) +$
$q_{05}\, q_{16}\, q_{23}\, q_{37} - q_{03}\, q_{16}\, q_{25}\, q_{37} - q_{05}\, q_{13}\, q_{26}\, q_{37} + q_{03}\, q_{15}\, q_{26}\, q_{37} +$

$q_{06}\,(q_{17}\,q_{23}\,q_{35} - q_{13}\,q_{27}\,q_{35} - \; q_{15}\,q_{23}\,q_{37} + q_{13}\,q_{25}\,q_{37})$,

$(M^{-1})^t_{31} = \frac{X_{31}}{Y_{31}}$ where
$X_{31} = q_{01}q_{17}q_{26}q_{35} - q_{01}q_{16}q_{27}q_{35} + q_{05}q_{17}q_{21}q_{36} - q_{01}q_{17}q_{25}\,q_{36} + q_{05}q_{27}q_{36} - q_{05}q_{11}q_{27}q_{36} + q_{01}q_{15}q_{27}q_{36} + q_{07}(q_{16}q_{21}q_{35} + q_{26}(q_{35} - q_{11}\,q_{35}) - (q_{15}\,q_{21} + q_{25} - q_{11}\,q_{25})\,q_{36}) - q_{05}\,q_{16}\,q_{21}\,q_{37} + q_{01}\,q_{16}\,q_{25}\,q_{37} - q_{05}\,q_{26}\,q_{37} + q_{05}\,q_{11}\,q_{26}\,q_{37} - q_{01}\,q_{15}\,q_{26}\,q_{37} + q_{06}\,(-(q_{17}\,q_{21}\,q_{35}) + (-1 + q_{11})\,q_{27}\,q_{35} + (q_{15}\,q_{21} + q_{25} - q_{11}\,q_{25})\,q_{37})$
$Y_{31} = -(q_{03}\,q_{17}\,q_{26}\,q_{35}) + q_{03}\,q_{16}\,q_{27}\,q_{35} - q_{05}\,q_{17}\,q_{23}\,q_{36} + q_{03}\,q_{17}\,q_{25}\,q_{36} + q_{05}\,q_{13}\,q_{27}\,q_{36} - q_{03}\,q_{15}\,q_{27}\,q_{36} + q_{07}\,(-(q_{16}\,q_{23}\,q_{35}) + q_{13}\,q_{26}\,q_{35} + q_{15}\,q_{23}\,q_{36} - q_{13}\,q_{25}\,q_{36}) + q_{05}\,q_{16}\,q_{23}\,q_{37} - q_{03}\,q_{16}\,q_{25}\,q_{37} - q_{05}\,q_{13}\,q_{26}\,q_{37} + q_{03}\,q_{15}\,q_{26}\,q_{37} + q_{06}\,(q_{17}\,q_{23}\,q_{35} - q_{13}\,q_{27}\,q_{35} - q_{15}\,q_{23}\,q_{37} + q_{13}\,q_{25}\,q_{37})$,

$(M^{-1})^t_{40} = (M^{-1})^t_{41} = (M^{-1})^t_{42} = (M^{-1})^t_{43} = 0, (M^{-1})^t_{44} = \frac{1}{24\,q_{44}}$,
$(M^{-1})^t_{45} = \frac{-13}{32\,q_{44}}, (M^{-1})^t_{46} = \frac{9}{16\,q_{44}}, (M^{-1})^t_{47} = \frac{-3}{8\,q_{44}}$

In the above expressions for $(M^{-1})^t$, the non-zero $q'_{ij}s,\ 0 \le i \le 4,\ 0 \le j \le 7$ are given in [8], e.g.

$$\begin{aligned}
q_{00} &= \frac{8\pi^3}{3}\left[ln\left(\frac{2^2.3^3}{6^6}\right) + i\pi\right] \\
q_{01} &= 2\pi^2\left[ln\left(\frac{2^2.3^3}{6^6}\right) + \frac{14}{3}\pi^2\right] \\
q_{02} &= \frac{4\pi^2}{\sqrt{3}}\left(\left[ln\left(\frac{2^2.3^3}{6^6}\right) + \frac{\pi}{\sqrt{3}}\right]^2 + 5\pi^2\right) \\
q_{03} &= \left(\left[ln\left(\frac{2^2.3^3}{6^6}\right) - \frac{\pi}{\sqrt{3}}\right]^2 + 5\pi^2\right) \\
q_{04} &= 4\sqrt{2}\pi\left(\left[ln\left(\frac{2^2.3^3}{6^6}\right) + i\pi\right]^4 + \frac{65}{3}\left[ln\left(\frac{2^2.3^3}{6^6}\right) + i\pi\right]^2 + \frac{169\pi^4}{3}\right. \\
&\left. -1440\left[ln\left(\frac{2^2.3^3}{6^6}\right) + i\pi\right] + 402\zeta(4)\right] \\
q_{05} &= 2\pi^2\left(\left[ln\left(\frac{2^2.33^3}{6^6}\right) + i\pi\right)^3 + 15\pi^2 ln\left(\frac{2^2.33^3}{6^6}\right) - 356\zeta(3)\right)
\end{aligned} \tag{98}$$

The matrix $(M^{-1})^t$ is non-singular as the determinant is non-zero. Using MATHEMATICA, one can actually evaluate T, but the expression is extremely long and complicated and will not be given in this paper.

The monodromy around $z = \infty$ can be evaluated as follows(similar to the way given in [35]). For

$|z| >> 1$, one can write:

$$\tilde{\Pi}_a(z) = \sum_{j=1}^{5} A_{aj}(z) u_j(z),\ \ a = 0, ..., 7, \tag{99}$$

where $u_j(z) = e^{-\frac{j}{6}}$. Now, as $z \to e^{2i\pi} z$, with obvious meanings to the notation:

$$T_u(\infty) = \begin{pmatrix} e^{-i\frac{\pi}{3}} & 0 & 0 & 0 & 0 \\ 0 & e^{-\frac{2i\pi}{3}} & 0 & 0 & 0 \\ 0 & 0 & e^{-i\pi} & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{4i\pi}{3}} & 0 \\ 0 & 0 & 0 & 0 & e^{-\frac{5i\pi}{3}} \end{pmatrix}. \tag{100}$$

Now, using

$$\tilde{\Pi}(z \to e^{2i\pi} z)|_{z\to\infty} = \mathcal{A}(z \to e^{2i\pi} z) T_u(\infty) u(z)|_{z\to\infty} \equiv T(\infty)\mathcal{A}(z) u(z)|_{z\to\infty}. \tag{101}$$

So, equation (101) is the defining equation for the monodromy matrix around $z \to \infty$. Note, however, that from the point of view of computations, given that the matrix A is not a square matrix, (101) involves solving 40 equations in 64 variables. The 8×5 matrix $A_{ai}(\infty)$ with $a = 0, ..., 7$ $i = 1, ..., 5$ for $z \to \infty$, is given below:

$$\mathcal{A}(\infty) = \\ \begin{pmatrix} -(2\pi)^{\frac{5}{2}} 6^{\frac{3}{2}} \frac{\sqrt{\pi}\Gamma(\frac{2}{3})}{\Gamma(\frac{7}{6})(\Gamma(\frac{5}{6})^2} & 54\sqrt{3}\pi^{\frac{3}{2}} \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{4}{3})\Gamma(\frac{2}{3})} & -288\sqrt{\frac{\pi}{3}} & -3\pi^2 \frac{\Gamma(\frac{7}{6})\Gamma(-\frac{1}{3})}{\Gamma(\frac{5}{3})\Gamma(\frac{1}{3})} & \frac{12\pi}{25\sqrt{3}} \frac{\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{6})}{\Gamma(\frac{11}{6})\Gamma(\frac{1}{6})} \\ \frac{216(\Gamma(\frac{2}{3})^2\Gamma(\frac{1}{3})}{(\Gamma(\frac{5}{6}))^2} & 0 & -\frac{16}{\sqrt{3}\pi} & 0 & -\frac{144\sqrt{3}\pi}{125} \frac{\Gamma(-\frac{2}{3})}{(\Gamma(\frac{1}{6}))^2} \\ \frac{216\pi\Gamma(\frac{2}{3})}{(\Gamma(\frac{5}{6}))^2} & 0 & 0 & \frac{-27\sqrt{\pi}\Gamma(\frac{1}{6})}{4(\Gamma(\frac{1}{3})^2} & \frac{36}{125}\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{6}) \\ \frac{216\Gamma(\frac{1}{6})\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{6})} & \frac{-27\pi\Gamma(-\frac{2}{3})}{8(\Gamma(\frac{1}{6}))^2} & 0 & -\frac{27\pi\Gamma(-\frac{2}{3})}{8(\Gamma(\frac{1}{6}))^2} & 0 \\ -216(\Gamma(\frac{1}{6}))^2\Gamma(\frac{2}{3}) & 0 & 0 & 0 & -\frac{216}{125}(\Gamma(\frac{5}{6}))^2\Gamma(-\frac{2}{3}) \\ -\frac{432\pi\Gamma(\frac{1}{6})}{\sqrt{3}\Gamma(\frac{5}{6})} & 0 & 2\sqrt{3}\pi & 0 & -\frac{216}{125}\frac{\Gamma(\frac{5}{6})\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{3})}{\Gamma(\frac{1}{6})} \\ \frac{216\pi\Gamma(\frac{1}{6})\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{6})} & 0 & 0 & -\frac{27\sqrt{\pi}}{4}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{1}{3})} & \frac{216}{125}\frac{\Gamma(\frac{5}{6})\Gamma(\frac{7}{6}\Gamma(-\frac{2}{3})\Gamma(-\frac{1}{6})}{\Gamma(\frac{1}{6})} \\ 216(\Gamma(\frac{1}{6}))^2\Gamma(\frac{2}{3}) & 27\sqrt{\pi}\Gamma(\frac{1}{3})\Gamma(-\frac{1}{6}) & 0 & 0 & -\frac{216}{125}\frac{\Gamma(\frac{5}{6})\Gamma(-\frac{2}{3})}{\Gamma(\frac{1}{6})} \end{pmatrix} \tag{102}$$

Given that $(T_u)_{ij} = e^{-\frac{\sqrt{-1}\pi i}{3}}\delta_{ij}$, no sum over i, one sees that the equation (101) becomes:

$$e^{-\frac{\sqrt{-1}\pi j}{3}} A_{aj}(\infty) = T_{ab}(\infty) A_{bj}(\infty), \tag{103}$$

(no sum over j) which needs to be solved for $T_{ab}(\infty)$. MATHEMATICA is unable to perform the required computation - however, it is in principle, doable.

Unfortunately, MATHEMATICA is not able to handle such a computation, this time. However, it is clear that it is in prinicple, a doable computation.

As done in [21], consider F-theory on an elliptically fibered Calabi-Yau 4-fold X_4 with holomorphic map $\pi : X_4 \rightarrow B_3$ and a 6-divisor D_3 as a section such that $\pi(D_3) = C_2 \subset B_3$. Then for vanishing size of the the elliptic fiber, it was argued in [21] that 5-branes wrapped around D_3 in M-theory on the same X_4 obeying the unit-arithmetic genus condition, $\chi(D_3, \mathcal{O}_{D_3}) = 1$, correspond to 3-branes wrapped around C_2 in type IIB, or equivalently F-theory 3-branes wrapped around $C_2 \subset B_3$. It was shown in [21] that only 3-branes contribute to the superpotential in F-theory. As there are no 3-branes in the F-theory dual [6], this implies that no superpotential is generated on the F-theory side. As F-theory 3-branes correspond to Heterotic instantons, one again expects no superpotential to be generated in Heterotic theory on the self-mirror $CY_3(11, 11)$ based on the $\mathcal{N} = 2$ type IIA/Heterotic dual of Ferrara et al where the same self-mirror Calabi-Yau figured on the type IIA side and the self-mirror nature was argued to show that there are no world-sheet or space-time instanton corrections to the classical moduli space.

If the abovementioned triality is correct, then one must be able to show that there is no superpotential generated on type IIA side on the freely-acting antiholomorphic involution of $CY_3(3, 243)$.

On the mirror type IIB side, the W is generated from domain-wall ($\equiv D5$-branes wrapped around supersymmetric 3-cycles $\hookrightarrow CY_3$'s) tention. $W_{IIB} = \int_{C:\partial C=\sum_i D_i} \Omega_3$, D_i's are 2-cycles corresponding to the positions of $D5$-branes or $O5$-planes, i.e., objects carrying $D5$ brane charge. From the world-sheet point of view, the $D5$ branes correspond to disc amplitudes and $O5$-planes correspond to $\mathbf{RP}^2$ amplitudes. As there are no branes in our theory, we need to consider only $\mathbf{RP}^2$ amplitudes. Now, type IIA on a freely acting involution of a Calabi-Yau with no branes or fluxes can still generate a superpotential because it is possible that free involution on type IIA side corresponds to orientifold planes in the mirror type IIB side, which can generate a superpotential.

The same can also be studied using localization techniques in unoriented closed string enumerative geometry [38]. Consider an orientation-reversing diffeomorphism $\sigma : \Sigma \rightarrow \Sigma$, an antiholomorhpic involution on the Calabi-Yau X $I : X \rightarrow X$ and an equivariant map $f : \Sigma \rightarrow X$ [satisfying $f \circ \sigma = I \circ f$]), then the quotient spaces in $\tilde{f} : \frac{\Sigma}{<\sigma>} \rightarrow \frac{X}{I}$ possesses a dianalytic structure. In the unoriented theory, one then has to sum over holomorphic and antiholomorphic instantons. For connected $\frac{\Sigma}{<\sigma>}$, the two contributions are the same; hence sufficient to consider only equivariant holomorphic maps. One constructs one-dimensional torus action, T, on X compatible with I with isolated fixed points. The action T induces an action on the moduli space of equivariant holomorphic maps, and one then evaluates the localized contributions from the fixed points, using an equivariant version of the the Atiyah-Bott formula, much on the lines of Kontsevich's work. For a Calabi-Yau 3-fold, the virtual cycle "$[\bar{M}_{g,0}(X, \beta)]^{virt}$" is zero-dimensional, and one has to evaluate $\int_{\Xi_s^{virt}} \frac{1}{e_T(N^{virt}_{\Xi_s})}$, where $\Xi_s \equiv$ is the fixed locus in the moduli space of symmetric holomorphic maps, and one sees that one gets a match with similar calculations based on large N dualities and mirror symmetry

For $\int d^2\theta W_{LG}$ to be invariant under $\Omega.\omega$, given that the measure is reflected under Ω, $\omega : W_{LG} \rightarrow -W_{LG}$.

away from the orbifold singularities: Promoting the action of ω to the one on the chiral superfields:

$$\omega : (\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_4, \mathcal{X}_5, \mathcal{X}_6, \mathcal{X}_0) \rightarrow (\bar{\mathcal{X}}_2, -\bar{\mathcal{X}}_1, \bar{\mathcal{X}}_4, \bar{\mathcal{X}}_5, \bar{\mathcal{X}}_6, \bar{\mathcal{X}}_0), \tag{104}$$

and using $Re(Y_i) = |\mathcal{X}_i|^2$, one gets the following action of ω on the twisted chiral superfields Y_i's:

$$\omega : Y_1 \rightarrow Y_2 + i\pi,\ Y_2 \rightarrow Y_1 + i\pi;\ Y_{0,4,5,6} \rightarrow Y_{0,4,5,6} + i\pi. \tag{105}$$

The action of ω on $Y_{4,5,6,0}$ implies that ω acts without fixed points even on the twisted chiral superfields, further implying that there are no orientifold fixed planes, and *hence no superpotential is generated on the type IIA side away from the orbifold singularities.*

<u>after singularity resolution</u>: Writing $W = \prod_{i=0}^{7} a_{e_{0,\ldots,7}} \mathcal{X}_i^{e_i}$ with the requirements that $\vec{l}^{(a)} \cdot \vec{e} = 0$ for $a = 1,2,3$ and $e_i \leq 1$[39][5], one sees that $\mathcal{X}_0 \prod_{i=1}^{7} \mathcal{X}_i$ is an allowed term in the superpotential. A valid antiholomorphic involution this time can be:

$$\omega : (\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_5, \mathcal{X}_6, \mathcal{X}_7) \rightarrow (\bar{\mathcal{X}}_0, \bar{\mathcal{X}}_2, -\bar{\mathcal{X}}_1, -\bar{\mathcal{X}}_3, \bar{\mathcal{X}}_4, \bar{\mathcal{X}}_5, \bar{\mathcal{X}}_6, \bar{\mathcal{X}}_7). \tag{106}$$

This on the mirror LG side again implies that one will have free actions w.r.t. $Y_{0,3,4,5,6,7}$ implying there can be no orientifold planes and *no superpotential (is likely to be) generated even after singularity resolution.*

The M-theory uplift of the type IIA side of the $\mathcal{N} = 1$ Heterotic/type IIA dual pair of [23], as obtained in [6], involves the 'barely G_2-manifold' $\frac{CY_3(3,243)\times S^1}{\mathbf{Z}_2}$. In this section we consider the $D = 11$ supergravity limit of M-theory and construct the $\mathcal{N} = 1, D = 4$ supergravity action, and evaluate the Kähler potential for the same.

The effect of the $\mathbf{Z}_2$ involution that reflects the S^1, $H^{1,1}(CY_3)$ and takes $H^{p,q}(CY_3)$ to $H^{q,p}(CY_3)$ for $p + q = 3$, where the CY_3 is the one that figures in $\frac{CY_3\times S^1}{\mathbf{Z}_2}$, at the level of $D = 11$ supergravity can be obtained by first compactifying the same on an S^1, then on CY_3 (following [40]) and eventually modding out the action by the abovementioned $\mathbf{Z}_2$ action.

The $D = 11$ supergravity action of Cremmer et al is:

$$\mathcal{L}_{11} = -\frac{1}{2} e_{11} R_{11} - \frac{1}{48}(G_{MNPQ})^2 + \frac{\sqrt{2}}{(12)^4}\epsilon^{M_0\ldots M_{10}} G_{M_0\ldots M_3} G_{M_4\ldots M_7} C^{11}_{M_8 M_9 M_{10}}, \tag{107}$$

which after dimensional reduction on an S^1, gives:

$$\begin{aligned}\mathcal{L}_{10} = &-\frac{1}{2} e_{10} R_{10} - \frac{1}{8} e_{10}\phi^{\frac{9}{4}} F^2_{mn} - \frac{9}{16} e_{10}(\partial_m ln\phi)^2 - \frac{1}{48} e_{10}\phi^{\frac{3}{4}}(F_{mnpq} + 6F_{[mn}B_{pq]})^2 \\ &-\frac{1}{12} e_{10}\phi^{-\frac{3}{2}} H^2_{mnp} + \frac{\sqrt{2}}{(48)^2}\epsilon^{m_0\ldots m_9}(F_{m_0\ldots m_3} + 6F_{m_0 m_1}B_{m_2 m_3})F_{m_4\ldots m_7}B_{m_8 m_9},\end{aligned} \tag{108}$$

where

$$\begin{aligned}&G_{MNPQ} = \partial_{[M} C_{NPQ]} \\ &F_{mnpq} = 4\partial_{[m} C_{npq]};\ B_{mn} = C_{mn10};\ H_{mnp} = 3\partial_{[m} B_{np]};\ F_{mn} = 2\partial_{[m} A_{n]}; \\ &C_{mnp} = A_{mnp} + 3A_{[m} B_{np]},\end{aligned} \tag{109}$$

and

$$\begin{aligned}&e_{11}{}^{A}_{\ M} = \begin{pmatrix} e_{10}{}^{a}_{\ m} & \phi A_M \\ 0 & \phi \end{pmatrix} \\ &A, M = 0, ..., 10;\ a, m = 0, ..., 9.\end{aligned} \tag{110}$$

[5]We thank A.Klemm for bringing [39] to our attention.

After compactifying on a CY_3, one gets the following Lagrangian density:

$$\mathcal{L}_4 = \mathcal{L}_4^{\mathrm{grav}+H^0} + \mathcal{L}_4^{H^2} + \mathcal{L}_4^{H^3}, \tag{111}$$

e.g., after a suitable Weyl scaling:

$$\mathcal{L}_4^{\mathrm{grav}} = e\Big[-\frac{R}{2} - \frac{G_{AB}}{2}\partial_\mu v^A\partial^\mu v^B - \frac{1}{4}\frac{\partial_\mu(\mathcal{V}\phi^{-3})^2}{(\mathcal{V}\phi^{-3})^2} + G_{\alpha\bar{\beta}}\partial_\mu Z^\alpha\partial^\mu \bar{Z}^\beta\Big],$$

where $A, B = 1, ..., h^{1,1}(CY_3)$, $\alpha, \beta = 1, ..., h^{2,1}(CY_3)$,

$$G_{AB} = \frac{i\int e^A \wedge *e^B}{2\mathcal{V}}, \mathcal{V} = \frac{1}{3!}\int J\wedge J\wedge J, G_{\alpha\bar{\beta}} = -\frac{i\int b_\alpha\wedge\bar{b}_\beta}{\mathcal{V}},$$
$$\bar{b}_{\alpha\ ij} = \frac{i}{||\Omega||^2}\Omega_i^{\bar{l}\bar{k}}\bar{\Phi}_{\alpha\ \bar{l}\bar{k}j}, \tag{112}$$

Φ being a (2,1) form, and the $h^{1,1}$ moduli $M^A = \sqrt{2}v^A\phi^{-\frac{3}{4}}$, entering in the variation of the metric with mixed indices and the $h^{2,1}$ moduli Z_α entering in the variation of the metric with same indices. one gets: Under the freely-acting antiholomorphic involution, the $h^{1,1}$-moduli M^A/v^A get projected out, G_{AB} is even, and A^A_μ gets projected out. Thus, one gets:

$$\mathcal{L}_4^{\mathrm{grav}}/\mathbf{Z}_2 = e\Big[-\frac{R}{2} - \frac{1}{4}\frac{(\partial_\mu(\mathcal{V}\phi^{-3}))^2}{(\mathcal{V}\phi^{-3})^2} + G_{\alpha\bar{\beta}}\partial_\mu Z^\alpha\partial^\mu\bar{Z}^\beta\Big]. \tag{113}$$

Defining $S \equiv \tilde{\phi} + iD - \frac{1}{4}(\Psi+\bar{\Psi})R^{-1}(\psi+\bar{\Psi})$, $\tilde{\phi} \equiv \sqrt{2}\mathcal{V}(v)\phi^{-3}$, $\Psi_{I(\equiv 0,1,...,h^{2,1})}$ appearing in the expansion of the real 3-form A_{mnp} in a canonical basis of H^3, D being a Lagrange multiplier, and

$$R_{IJ} \equiv Re[\mathcal{N}_{IJ}],\ \mathcal{N}_{IJ} \equiv \frac{1}{4}\bar{F}_{IJ} - \frac{(NZ)_I(NZ)_J}{(ZNZ)},\ (R^{-1})^{IJ} = 2\Big(N^{-1}(\mathbf{1} - \bar{K}\bar{Z} - KZ)\Big)^{IJ}, \tag{114}$$

where Z^I and iF_I are the period integrals, $N_{IJ} = \frac{1}{4}(F_{IJ} + \bar{F}_{IJ})$, and $K_I \equiv \frac{\int\Omega_I\wedge\bar{\Omega}}{\int\Omega\wedge\bar{\Omega}}$, with $\Omega_I \equiv \frac{\partial\Omega}{\partial Z^I}$. Here, it is assumed that the holomorphic 3-form Ω is expaneded in a canonical cohomology basis (α_I, β^I) satisfying

$$\int_{A^J}\alpha_I = \int\alpha_I\wedge\beta^J = -\int_{B_I}\beta^J = -\int\beta^J\wedge\alpha_I = \delta_I^J, \tag{115}$$

with (A^I, B_I) being the dual homology basis. The period integrals are then defined to be: $Z^I = \int_{A^I}\Omega$ and $iF_I = \int_{B_I}\Omega$. Hence,

$$\Omega = Z^I\alpha_I + iF_I\beta^I. \tag{116}$$

For the $\mathcal{N} = 1$ case, we work in the large volume limit of the Calabi-Yau. In this limit, one gets:

$$\mathcal{L}^{\mathrm{grav}+H^0+H^2+H^3}/\mathbf{Z}_2 = e\Big[-\frac{R}{2} - G_{AB}\partial_\mu a^A\partial^\mu a^B + G_{\alpha\bar{\beta}}\partial_\mu Z^\alpha\partial^\mu Z^{\bar{\beta}} - \frac{1}{2}\frac{(\partial_\mu\tilde{\phi})^2}{\tilde{\phi}^2}\Big]. \tag{117}$$

Hence, one gets for the $\mathcal{N}=1$ Kähler potential $K_{\mathcal{N}=1}$:

$$K_{\mathcal{N}=1} = K[a^A, Z^\alpha] + \frac{1}{2} ln[\tilde{\phi}]. \tag{118}$$

At the $\mathcal{N}=2$ level, if there were decoupling between the fields of the H^3-sector from the other fields of the other sectors, the Kähler potential would be consisting of $ln\Big[S+\bar{S}+\frac{1}{2}(\Psi+\bar{\Psi})R^{-1}(\Psi+\bar{\psi})\Big]$,[6] from the H^3-sector. From the definitin of S above, one sees that:

$$S + \bar{S} + \frac{1}{2}(\Psi+\bar{\Psi})R^{-1}(\Psi+\bar{\Psi}) = 2\tilde{\phi}. \tag{119}$$

This partially explains the appearance of $ln[\tilde{\phi}]$ in $K_{\mathcal{N}=1}$.

Given the action of an antiholomorphic involution on the cohomology, it is in general quite non-trivial to figure out the action of the involution on the period integrals using the canonical (co)homology basis of (115). We now discuss a reasonable guess for the same. From (119), one sees that the RHS is reflected under the antiholomorphic involution discussed towards the beginning of this section. We now conjecture that on the LHS, this would imply that

$$S \rightarrow -S,\ (\Psi+\bar{\Psi})^2 \rightarrow (\Psi+\bar{\Psi})^2, R_{IJ} \rightarrow -R_{IJ}. \tag{120}$$

We further conjecture that $R_{IJ} \rightarrow -R_{IJ}$ is realized by

$$Z^I \rightarrow -\bar{Z}^I,\ iF_I \rightarrow -i\bar{F}_I\ \alpha_I \rightarrow -\alpha_I,\ \beta^I \rightarrow \beta^I. \tag{121}$$

One should note that given that the antiholomorphic involution is orientation reversing, the intersection form $\int \alpha_I \wedge \beta^J$ is also reflected. This fact, e.g., can be explicitly seen in the real basis of $H^3(T^6, \mathbf{Z})$[41, 8]: Further, the conjecture at the level of action on the cohomology, can also be checked for the mirror to the quintic, for which $h^{2,1}=1$.

4 Connection with MQCD

In this section, we make some observations and speculative remarks regarding the possible relationship between each of the two parts of the paper above and Witten's MQCD. When uplifting configurations involving $NS5$ and $D4$ branes to M-theory that displayed the interesting properties of chiral symmetry breaking, confinement, etc, Witten in [34] showed that the world volume of the $M5$-brane is topologically $R^{1,3}\times\Sigma$, where Σ is a Riemann surface (embedded in a Calabi-Yau 3-fold for $N=1$ MQCD). Now, the compact $CY_3(3,243)$ has been central in the second part of this paper, both at the $N=2$ and at the $N=1$ levels. The mirror to the same, $CY_3(243,3)$ as shown in [42], can be expressed as a $K3$-fibration over $\mathbf{CP}^1$ and written as the following degree-24 hypersurface in $\mathbf{WCP}^4[1,1,2,8,12]$:

$$W(a,b,c) \equiv \frac{1}{24}(\zeta + \frac{b}{\zeta} + 2)x_0^{12} + \frac{1}{12}x_3^12 + \frac{1}{3}x_4^3 + \frac{1}{2}x_5^3 + \frac{1}{6\sqrt{c}}(x_0x_3)^6 + (\frac{a}{\sqrt{c}})^{\frac{1}{6}} x_0x_3x_4x_5 = 0, \tag{122}$$

[6] This however assumes that $\frac{\partial \bar{\mathcal{N}}_{IJ}}{\partial z^K} = 0 \leftrightarrow \frac{1}{4}F_{IJK} - \frac{(\frac{1}{4}F_{IRK}\bar{z}^R(N\bar{Z})_J + \frac{1}{4}(N\bar{Z}_I F_{JLK}\bar{Z}^L)}{(\bar{Z}N\bar{Z})} + \frac{(N\bar{Z})_I(N\bar{Z})_J}{(2\bar{Z}N\bar{Z})^2}\bar{Z}^P\bar{Z}^Q F_{PQK} = 0$

where $a \equiv -\frac{\psi_0^6}{\psi_1}$, $b \equiv \psi_2^{-2}$, $c \equiv -\frac{\psi_2}{\psi_1^2}$, $\psi_{0,1,2} \equiv$ three complex structure deformation parameters entering $W(x_0, x_1, x_2, x_3, x_4, x_5; \psi_0, \psi_2, \psi_2)$. Further, in the $x_2 \neq 0$ coordinate patch, $\frac{x_1}{x_2} \equiv \frac{\zeta^{\frac{1}{12}}}{b^{\frac{1}{24}}}$ and $x_1^2 \equiv x_0 \zeta^{\frac{1}{12}}$, ζ being $\mathbf{CP}^1$ coordinate. By choosing a particular α'-scaling for the three complex structure deformation parameters:

$$a \sim \epsilon,\ b \sim \epsilon^2, 1 - c \sim \epsilon, \epsilon \equiv (\alpha')^{\frac{3}{2}} \to 0, \tag{123}$$

to get the $SU(3)$ Seiberg-Witten regime, , the corresponding hypersurface can be rewritten

$$\begin{aligned}&(\alpha')^{\frac{3}{2}}(z + \frac{\Lambda^6}{z} + 2P_{A_2}(x, u, v) + y^2 + w^2) + O(\epsilon^2) \\ &= \epsilon^{\frac{3}{2}}(\prod_{i=1}^{3}(x - a_i(z)) + y^2 + w^2) + O(\epsilon^2),\end{aligned} \tag{124}$$

where one sees the Riemann surface [42]

$$\Sigma : \prod_{i=1}^{3}(x - a_i(z)). \tag{125}$$

In addition, chiral symmetry breaking in the model results in the formation of domain wall separating different vacua, whose world-volume is topologically given by $R^{1,2}(X^{0,1,2} \times S(x^{3,4,5})$, where S is a supersymmetric 3-cycle embedded in a G_2-manifold that is topologically $\mathbf{R}(x^3) \times (\mathbf{R}^5(x^{4,5,6,7,8} \times S^1(x^{10})$ Complexifying the coordinates, $v = x^4 + ix^5, w = x^7 + ix^8, s = x^6 + ix^8, t = e^{-s}$, the boundary condition on S is that as $x^3 \to -\infty$, $S \to \mathbf{R} \times \Sigma$ and as $x^3 \to \infty$, $S \to \mathbf{R} \times \Sigma'$, where $\Sigma : w = \zeta v^{-1},\ t = v^n$ and $\Sigma' : w = e^{\frac{2\pi i}{n}} \zeta v^{-1}, t = v^n$. Following the notations of [43], the calibration for G_2 manifolds can be written as: $\Phi = e^{123} + e^{136} + e^{145} + e^{235} - e^{246} + e^{347} + e^{567}$ (slightly different from (20); $e^{ijk} \equiv e^i \wedge e^j \wedge e^k$), and then the supersymmetric 3-cycle embedded in the G_2-manifold will be given as: $w = w(x^3, v, \bar{v}), s = s(x^3, v, \bar{v})$. Then with the choice of vielbeins as:$e^1 = dx^{10}, e^2 = dx^5, e^3 = dx^3, e^4 = dx^7, e^5 = dx^4, e^6 = dx^6, e^7 = dx^8$ and $x^6 = A, x^7 = C, x^8 = D, x^{10} = B$, the condition for supersymmetric cycle: $\Phi|S = \sqrt{g} dx^3 \wedge dx^4 \wedge dx^5$, after further relabeling $x^{3,4,5}$ as $y^{3,1,2}$ and after assuming: $\partial_1 A = \partial_2 B,\ \partial_2 A = -\partial_1 B (\equiv$ Cauchy-Riemann condition), translates to give:

$$\begin{aligned}&[\partial_1 A \partial_3 A - \partial_2 \partial_3 B + \partial_1 C \partial_3 C - \partial_2 C \partial_3 D]^2 + [\partial_2 A \partial_3 A + \partial_1 A \partial_3 B + \partial_2 C \partial_3 C + \partial_1 C \partial_3 D]^2 \\ &= [1 + (\partial_1 A)^2 + (\partial_2 A)^2 + (\partial_1 C)^2 + (\partial_2 C)^2][(\partial_3 A)^2 + (\partial_3 B)^2 + (\partial_3 C)^2 + (\partial_3 D)^2].\end{aligned} \tag{126}$$

The ansatz to solve (126) taken in [43] was:

$$\begin{aligned}A(y_1, y_2, y_3) &= -ln(y_1^2 + y_2^2) + \sum_{m=1}^{\infty} \frac{2}{coshy_3}^{2m} a_{2m}, \\ B(y_1, y_2, y_3) &= -2tan^{-1}(\frac{y_2}{y_1}) + \sum_{m=1}^{\infty} \frac{2}{coshy_3}^{2m} b_{2m}, \\ C(y_1, y_2, y_3) &= (tanhy_3)(\frac{-y_1 \zeta}{y_1^2 + y_2^2}), \\ D(y_1, y_2, y_3) &= (tanhy_3)(\frac{y_2 \zeta}{y_1^2 + y_2^2}).\end{aligned} \tag{127}$$

However, no explicit forms of a_{2m} and b_{2m} were given that would solve (126). In [4], however, choosing: $e^1 = dx^{10}$, $e^2 = dx^5$, $e^3 = dx^3$, $e^4 = dx^7$, $e^5 = dx^6$, dx^8, and for the $SU(2)$ embedding of the supersymmetric 3-cycle in the G_2-manifold, the ansatz taken is:

$$\begin{aligned}
v &= \left[e^{\frac{y_1}{2}} + \sum_{m=1}^{\infty}\left(\frac{1}{2coshy_3}\right)^{2m} f_{2m}(y_1)\right]e^{iy_2},\\
w &= -\zeta tanhy_3\left[e^{-\frac{y_1}{2}} + \sum_{m=1}^{\infty}\left(\frac{1}{2coshy_3}\right)^{2m} g_{2m}(y_1)\right]e^{-iy_1},\\
s &= -y_1 - \sum_{m=1}^{\infty}\left(\frac{1}{2coshy_3}\right)^{2m} h_{2m}(y_1) - 2iy_2,
\end{aligned} \tag{128}$$

where for the $SU(2)$ group, f_{2m}, g_{2m}, h_{2m} can be complex, but are taken to be real in [4]. The condition for getting a supersymmetric 3-cycle implemented by ensuring that the pull-back of the calibration Φ to the world volume of the 3-cycle is identical to the volume form on the 3-cycle, gives recursion relations between the coefficients f_{2m} and g_{2m}, by setting $h_{2m} = 0$, e.g. for $m = 1$, as shown in [4],

$$\begin{aligned}
&-\zeta e^{-\frac{y_1}{2}}\partial_1 f_2 + (2e^{\frac{y_1}{2}} + \frac{\zeta}{2}e^{-\frac{y_1}{2}}f_2 - \zeta e^{\frac{y_1}{2}}\partial_1 g_2 - (2\zeta^2 e^{\frac{-y_1}{2}} + \frac{\zeta}{2}e^{\frac{y_1}{2}})g_2 = -4\zeta^2 e^{\frac{-y_1}{2}},\\
&-(\zeta^2 e^{-y_1} + 4)f_2 + 2\zeta\partial_1 g_2 - (\zeta^2 - \zeta)g_2 = -2\zeta^2 e^{\frac{-y_1}{2}}.
\end{aligned} \tag{129}$$

One can subsitute for f_2 from the second equation and get a second order differential equation for g_2. However, it is shown that in the limit $\zeta \to 0$, one can consistently set $f_{2m} = h_{2m} = 0, m \geq 1$. Further, surprisingly, as perhaps missed to be noticed in [4], one also gets the following differential equation for all g_{2m}'s, $m \geq 1$:

$$2\partial_1 g_{2m} + g_{2m} = \mathcal{O}(\zeta) \to 0, \tag{130}$$

implying

$$g_{2m} = e^{\frac{-y_1}{2}},\ m \geq 1. \tag{131}$$

Hence, (128) becomes:

$$\begin{aligned}
v(y_1, y_3) &= e^{\frac{y_1}{2+iy_2}},\\
w(y_1, y_3) &= -\zeta\frac{tanh(y_3)e^{\frac{-y_1}{2}}}{1 - (\frac{sech(y_3)}{2})^2}e^{-iy_2},\\
s(y_1, y_3) &= -y_1 - 2iy_2.
\end{aligned} \tag{132}$$

One thus gets a convergent solution, unlike the case for finite ζ as pointed out in [44].

Also, Seiberg-Witten equations for $N = 2$ MQCD are extensively used in the analysis. Now, by compactifying $E_8 \times E_8$ Heterotic string theory on a T^2 at the complex structure modulus equal to the Kähler modulus equal to i, there is an enhanced $SU(2) \times SU(2)$ gauge symmetry and by embedding of $SU(2)$ gauge bundles in the two E_8's and one of the two $SU(2)$'s and subsequent Higgsing away of the resultant $E_7 \times E_7$, one gets the (Heterotic) string analog of $N = 2$ Seiberg-Witten theory [26].

Hence, we see a connection between the first two parts of this paper and MQCD via the existence of supersymmetric 3-cycles embedded in G_2-manifolds (around which $M2$ branes wrapped to produce membrane instantons), and the Riemann surface that is common to both the world-volume of the $M5$ brane used to reproduce the type IIA brane configurations, and the compact Calabi-Yau $CY_3(3,243)$ that appeared ubiquitously in Section **3**.

5 Conclusion and Discussions

In this paper, we have evaluated in a closed form, the exact expression for the nonperturbative contribution to the superpotential from a single $M2$-brane wrapping an isolated supersymmetric 3-cycle of a G_2-holonomy manifold. The comparison with Harvey and Moore's result, is suggestive but not exact. A heat-kernel asymptotics analysis for a non-compact smooth G_2-holonomy manifold that is metrically $\mathbf{R}^4 \times T^3$, in the adiabatic approximation, showed that the UV-divergent terms of one of the bosonic and the fermionic determinants are proportional to each other, to the order we calculate, indicative of cancellation between the same, as expected because the $M2$ brane action of Bergshoeff, Sezgin and Townsend is supersymmetric. *Unlike the result of [13], the expression obtained for the superpotential above in terms of fermionic and bosonic determinants, in addition to a holomorphic phase factor, is valid even for non-rigid supersymmetric 3-cycles as the one considered in (18) above.* Following Gubser et al in [10], it is tempting to conjecture that the superpotential term corresponding to multiple wrappings of the $M2$-brane around the supersymmetric 3-cycle, should be give by:

$$\Delta W = \sum_n \sqrt{\frac{det\mathcal{O}_3}{\det\mathcal{O}_1 det\mathcal{O}_2}} \frac{e^{n\int_\Sigma [iC - \frac{1}{l_{11}^3} vol(h)]}}{n^2}. \tag{133}$$

In terms of relating the result obtained in (9) to that of the 1-loop Schwinger computation of M theory and the large N-limit of the partition function evaluated in [45][7], one notes that the 1-loop Schwinger computation also has as its starting point, an infinite dimensional bosonic determinant of the type $det\Bigl((i\partial - eA)^2 - Z^2\Bigr)$, A being the gauge field corresponding to an external self-dual field strength, and Z denoting the central charge. The large N-limit of the partition function of Chern Simons theory on an S^3, as first given by Periwal in [46], involves the product of infinite number of sin's, that can be treated as the eigenvalues of an infinite determinant. This is indicative of a possible connection between the membrane instanton contribution to the superpotential, the 1-loop Schwinger computation and the large N limit of the Chern-Simons theory on an S^3. Also, there were interesting similarities and differences between the membrane instanton result of Section **2** and Witten's heterotic world-sheet instanton result in terms of the forms of expressions and the boson-fermion determinant cancellation in both.

We also related the $\mathcal{N}=1$ Heterotic theory on a self-mirror CY_3 to the nonperturbative formulations of type IIA and IIB, namely M and F theories. While on the M-theory side, the suitable manifold turned out to one of $SU(3)\times\mathbf{Z}_2$ holonomy, referred to as a 'barely G_2 manifold', the elliptically fibered Calabi-Yau 4-fold involves a trivial $\mathbf{CP}^1$-fibration over the Enriques surface for its base, and surprisingly has a Hodge data that can not be obtained as a free involution of ($\mathcal{N}=2$ F-theory on) $CY_3(3,243)\times T^2$.

[7]This logic was suggested to us by R.Gopakumar.

The precise construction of the CY_4 used in the F-theory dual and its connection with the $\mathcal{N}=2$ parent model of F-theory on $CY_3(3,243) \times T^2$, needs to be understood.

We also obtained the Meijer basis of solution to the Picard-Fuchs equation for the Landau-Ginsburg model corresponding to $CY_3(3,243)$ *after* the resolution of the orbifold singularities of the degree-24 Fermat hypersurface in $\mathbf{WCP}^4[1,1,2,8,12]$, in the large *and* small complex structure limits, getting the ln-terms without resorting to parametric differentiations of infinite series. We also discussed in detail the evaluation of the monodromy matrix in the large complex structure limit. We also considered the action of an antiholomorphic involution on $D=11$ supergravity compactified on $CY_3(3,243) \times S^1$, and evaluated the form of the $\mathcal{N}=1$ Kähler potential. In the process, we also gave a conjecture on the action of the involution on the periods of $CY_3(3,243)$, given its action on the cohomology of the same. We verified the conjecture for T^6 for the periods and cohomology basis, and for the mirror to the quintic for the cohomology basis. Finally, we showed that no superpotential is generated in type IIA and hence M-theory sides using mirror symmetry, *after* the resolution of the orbifold singularities associated with the Fermat hypersurface whose blow up gives $CY_3(3,243)$.

The reason for considering membrane instantons involving $M2$ branes wrapping around supersymmetric 3-cycles embedded in G_2-holonomy manifold and topics related to the compact Calabi-Yau $CY_3(3,243)$ in the same article was because we have tried to attempt to make an indirect connection between these two topics by relating both of them individually to Witten's MQCD. The latter involves uplifting suitable configurations of $NS5$ and $D4$ branes to M theory involving $M5$ branes with a suitable topology, as well as domain walls being modelled again by $M5$ branes with a particular topology, the former case consisting of a Riemann surface that is also present in the defining hypersurface of $CY_3(3,243)$, and the latter corresponding to precisely a supersymmetric 3-cycle embedded in a G_2-holonomy manifold. We also showed that in the limit of vanishing of a certain complex constant that figures in the Riemann surface when referring to the boundary conditions satisfied by the supersymmetric 3-cycle embedded in G_2-manifold, it was possible to get an exact answer for the embedding, using the ansatz of [4].

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